Lecture Notes for

Logic and Computability

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Natural Deduction for Propositional Logic

We started the first chapter by using our common sense to conclude new knowledge from given knowledge (we concluded that "there were taxis at the airport" and that "John has his sun creme with him"). Our goal is to perform this reasoning formally and automatically. Natural deduction is a calculus for reasoning about propositions so that we can establish the validity of arguments. Therefore, natural deduction defines a *set of rules* each of which allows us to draw a conclusion given a certain arrangement of premises. By successively applying these rules, we are able to *infer* a conclusion from a set of premises.

Sequents

Our goal is to apply proof rules to a set of given formulas, *the premises* to eventually obtain a new formula, *the conclusion*. Formally, we write:

$$\varphi_1, \varphi_2, \ldots, \varphi_n \;\vdash\; \psi$$

We call this expression a *sequent*. We say that the *premises* (formulas on the left side) *entail* the *conclusion* (formulas on the right). *A sequent is valid if a proof for it can be found.*

Example. The sequent for the motivating example in Chapter 1 is:

$$p \wedge \neg q \to r,\, \neg r,\, p \;\vdash\; q$$

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4.1 Rules for Natural Deduction

For each of the connectives, there is one or more rules to introduce it and one or more rules to eliminate it.

The AND-Introduction Rule

First, we consider the rule for introducing a conjunction, called the *and-introduction-rule.* Given the two premises φ and ψ, the rule allows us to conclude $\varphi \land \psi$. We write:

$$\frac{\varphi \qquad \psi}{\varphi \land \psi} \land_i$$

Above the line we write the two premises φ and ψ of the rule. Below the line we write the conclusion $\varphi \land \psi$. To the right of the line, we state the name of the rule; and-introduction is abbreviated by $\land_i$.

The intuition of the rule is the following: If we have two formulas that are known to be true separately (the premises), then we can conclude that the conjunction of the two premises must also be true.

Construction of a Natural Deduction Proof

Next, we discuss how to construct a proof using the natural deduction rules to show that a given sequent is valid.

Exercise 4.1

Give the proof for the sequents $p, q \vdash p \land q$ and $p, q \vdash q \land p$.

Solution.

$p, q \vdash p \land q$

1.	p	premise
2.	q	premise
3.	$p \land q$	$\land_i 1, 2$

$p, q \vdash q \land p$

1.	p	premise
2.	q	premise
3.	$q \land p$	$\land_i 2, 1$

Each line of the proof consists of the *line number*, a *formula*, and *the reason for having the formula.* We start the proof by writing down the premises, leaving a gap, and writing the conclusion in the end. The task is to apply the rules such that we fill the gap. In this case, we only need to write down, that we applied the $\land_i$ rule, once combining line 1 and line 2, and once in the reverse order, to justify the conclusion.

The AND-Elimination Rule

Given the premise $\varphi \land \psi$, the elimination rules allows us to conclude φ as well as ψ. We write:

$$\frac{\varphi \land \psi}{\varphi} \land_{e_1} \qquad \frac{\varphi \land \psi}{\psi} \land_{e_2}$$

The rule $\land_{e_1}$ is used to derive the first subformula, the rule $\land_{e_2}$ is used to derive the second sub-formula. Intuitively, if a conjunction is known to be true, each sub-formula must also be true. Intuitive illustration:

- The earth is a planet and the sun is a star. (Premise)
- Therefore: The earth is a planet. ($\land_{e_1}$ line 1)
- Therefore: The sun is a star. ($\land_{e_2}$ line 1)

Exercise 4.2

Give the proof for the sequents $p \land q \vdash p$ and $p \land q \vdash q$.

Solution.

$p \land q \vdash p$

1.	$p \land q$	premise
2.	p	$\land_{e_1} 1$

$p \land q \vdash q$

1.	$p \land q$	premise
2.	q	$\land_{e_2} 1$

Exercise 4.3

Give the proof for the sequent $p \land q, r \vdash q \land r$.

Solution.

1.	$p \land q$	premise
2.	r	premise
3.	q	$\land_{e_2} 1$
4.	$q \land r$	$\land_i 3, 2$

Exercise 4.4

Give the proof for the sequent $(p \wedge q) \wedge r, s \wedge t \;\vdash\; q \wedge s$.

Solution.

1.	$(p \wedge q) \wedge r$	premise
2.	$s \wedge t$	premise
3.	$p \wedge q$	$\wedge_{e_1} 1$
4.	q	$\wedge_{e_2} 3$
5.	s	$\wedge_{e_1} 2$
6.	$q \wedge s$	$\wedge_i 4, 5$

In order to form the conclusion, the propositions q and s are needed. q can be extracted from the first premise. Note: a natural deduction rule can only be applied on the *top-level connective* of a formula. Hence, we need to apply the $\wedge_e$ rule once to get $p \wedge q$, and then a second time to get q. Furthermore, we need the propositional atom s to form the conclusion. We get s from the second premise, again, by using the $\wedge_e$ rule. Finally, q and s can be connected using the $\wedge_i$ rule to form the conclusion.

The Double-Negation-Introduction Rule

If a formula φ holds, also $\neg\neg\varphi$ must be true, since they are equivalent. The rule looks as following.

$$\frac{\varphi}{\neg\neg\varphi}\ \neg\neg_i$$

Intuitively, the sentence “The ocean is salty” is the same as saying “It is not true that the ocean is not salty.”

Exercise 4.5

Give the proof for the sequent $p \wedge q \;\vdash\; \neg\neg p$.

Solution.

1.	$p \wedge q$	premise
2.	p	$\wedge_{e_1} 1$
3.	$\neg\neg p$	$\neg\neg_i 2$

The Double-Negation-Elimination Rule

The rule is written as follows.

$$\frac{\neg\neg\varphi}{\varphi}\ \neg\neg_e$$

Same argument as before, the two formulas are equivalent. If it is true that "Great Britain is not not a monarchy", then we can follow that "Great Britain is a monarchy".

Exercise 4.6

Give the proof for the sequent $\neg\neg p \wedge \neg\neg q \ \vdash \ p \wedge q$.

Solution.

1.	$\neg\neg p \wedge \neg\neg q$	premise
2.	$\neg\neg p$	$\wedge_{e_1} 1$
3.	$\neg\neg q$	$\wedge_{e_2} 1$
4.	p	$\neg\neg_e 2$
5.	q	$\neg\neg_e 3$
6.	$\wedge q$	$\wedge_i 4, 5$

Exercise 4.7

Give the proof for the sequent $p, \neg\neg(q \wedge r) \ \vdash \ \neg\neg p \wedge r$.

Solution.

1.	p	prem.
2.	$\neg\neg(q \wedge r)$	prem.
3.	$\neg\neg p$	$\neg\neg_i 1$
4.	$q \wedge r$	$\neg\neg_e 2$
5.	r	$\wedge_{e_2} 4$
6.	$\neg\neg p \wedge r$	$\wedge_i 3, 5$

The Implication-Elimination Rule

The implication-elimination rule states that, if we know that φ holds and we know that $\varphi \to \psi$, we can conclude that ψ holds.

$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \to_e$$

Intuitively, if we know that it is true that "It is snowing", and "If it is snowing then it is cold", then we can conclude that "It is cold".

Exercise 4.8

Give the proof for the sequent $\neg\neg p, p \to q \;\vdash\; \neg\neg q$.

Solution.

1.	$\neg\neg p$	premise
2.	$p \to q$	premise
3.	p	$\neg\neg_e 1$
4.	q	$\to_e 1, 2$
5.	$\neg\neg q$	$\neg\neg_i 4$

Exercise 4.9

Give the proof for the sequent $p \wedge \neg a, p \wedge \neg a \to q \vee b \;\vdash\; q \vee b$.

Solution.

1.	$p \wedge \neg a$	premise
2.	$p \wedge \neg a \to q \vee b$	premise
3.	$q \vee b$	$\to_e 1, 2$

Exercise 4.10

Give the proof for the sequent $p, p \to q, p \to (q \to r) \;\vdash\; r$.

Solution.

1.	p	premise
2.	$p \to q$	premise
3.	$p \to (q \to r)$	premise
4.	$q \to r$	$\to_e 1, 3$
5.	q	$\to_e 1, 2$
6.	r	$\to_e 4, 5$

The Modus-Tollens Rule (MT)

Before discussing the implication-introduction rule, let us consider a derived rule from the implication-elimination rule called modus tollens. If it holds that $\varphi \to \psi$ and $\neg\psi$ are true, then we can conclude $\neg\varphi$.

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi}\ \text{MT}$$

Intuitive argumentation. The following is true: "If the sun is shining it is daytime" and "It is not daytime". Therefore, we can conclude using modus tollens that "The sun is not shining".

Exercise 4.11

Give the proof for the sequent $\neg p \to q, \neg q \ \vdash \ p$.

Solution.

1.	$\neg p \to q$	premise
2.	$\neg q$	premise
3.	$\neg\neg p$	MT $1, 2$
4.	p	$\neg\neg_e 3$

Exercise 4.12

Give the proof for the sequent $\neg p \to (q \to r), \neg p, \neg r \ \vdash \ \neg q$.

Solution.

1.	$\neg p \to (q \to r)$	premise
2.	$\neg p$	premise
3.	$\neg r$	premise
4.	$q \to r$	$\to_e$ $1, 2$
5.	$\neg q$	MT $4, 3$

The Implication-Introduction Rule

The $\to$i rule says that in order to prove $\varphi \to \psi$, we make a temporary assumption φ and then prove ψ. The rule is formally written as:

$$\frac{\boxed{\begin{array}{l}\varphi \text{ ass.}\\ \vdots \\ \psi\end{array}}}{\varphi \to \psi} \to_i$$

Let's assume that we want to prove the sequent $p \to q,\ q \to r \ \vdash \ p \to r$. To prove this sequent, we *temporarily assume* that p holds. *Under the assumption* that p holds, we can derive from the first premise that q holds, and using q we can derive that r holds from the second premise. Thus, by *assuming* that p holds, we can *imply* that r holds, which we express symbolically by $p \to r$. The proof is given below.

1.	$p \to q$	premise
2.	$q \to r$	premise
3.	p	assumption
4.	q	$\to_e$ 3, 1
5.	r	$\to_e$ 4, 2
6.	$p \to r$	$\to_i$ 3 − 5

The *assumption box* in the proof defines the scope of the temporary assumption p. By applying other rules, we can derive new formulas within the box. But everything that we derive inside of the box still depends on the assumption of p. Only by applying the $\to_i$ rule are we allowed to conclude $p \to r$. We will introduce additional rules that use boxes. *It is important that the line immediately following a closed box has to match the pattern of the conclusion of the rule that uses the box.* For the $\to_i$ rule this means that we have to continue after the box with $\varphi \to \psi$. Within the box, φ is the formula in the first line and ψ the formula of the last line.

Exercise 4.13

Give the proof for the sequent $p \to q \ \vdash \ \neg q \to \neg p$.

Solution.

1.	$p \to q$	premise
2.	$\neg q$	assumption
3.	$\neg p$	MT 1, 2
4.	$\neg q \to \neg p$	$\to_i$ 2 − 3

Exercise 4.14

Give the proof for the sequent $\neg q \to \neg p \;\vdash\; p \to \neg\neg q$.

Solution.

1.	$\neg q \to \neg p$	premise
2.	p	assumption
3.	$\neg\neg p$	$\neg\neg_i 2$
4.	$\neg\neg q$	MT 1,3
5.	$p \to \neg\neg q$	$\to_i 2-4$

Exercise 4.15

Give the proof for the sequent $p \wedge q \to r \;\vdash\; p \to (q \to r)$.

Solution.

1.	$p \wedge q \to r$	premise
2.	p	assumption
3.	q	assumption
4.	$p \wedge q$	$\wedge_i 2, 3$
5.	r	$\to_e 4, 1$
6.	$q \to r$	$\to_i 3-5$
7.	$p \to (q \to r)$	$\to_i 2-6$

Exercise 4.16

Give the proof for the sequent $p \to (q \to r) \;\vdash\; p \wedge q \to r$.

Solution.

1.	$p \to (q \to r)$	premise
2.	$p \wedge q$	assumption
3.	p	$\wedge_{e_1} 2$
4.	q	$\wedge_{e_2} 2$
5.	$q \to r$	$\to_e 3, 1$
6.	r	$\to_e 4, 5$
7.	$p \wedge q \to r$	$\to_i 2-6$

The Disjunction-Introduction Rule

If we know that φ holds, we can derive that $\varphi \lor \psi$ holds and that $\psi \lor \varphi$ holds. This is true for any ψ. The rule is formulated as follows:

Formally the rules are written as:

$$\frac{\varphi}{\varphi \lor \psi} \lor_{i_1} \qquad \frac{\varphi}{\psi \lor \varphi} \lor_{i_2}$$

Exercise 4.17

Give the proofs for the sequent $p \ \vdash \ (q \to r \land s) \lor p$.

Solution.

1.	p	premise
2.	$(q \to r \land s) \lor p$	$\lor_{i_2} 1$

The Disjunction-Elimination Rule

From a given formula $\varphi \lor \psi$, we want to proof some other formula χ. We only know that φ *or* ψ holds. It could be that both of them are true, but it could also be that only ψ is true, or only φ is true. Sine we don't know which sub-formula is true, we have to give two separate proofs:

- First box: We assume φ is true and need to find a proof for χ.
- Second box: We assume ψ is true and need to find a proof for χ.

Only if we can prove χ in the first and in the second box, then we can conclude that χ holds also outside of the box.

The $\lor_e$ rules says that we can only derive χ from $\varphi \lor \psi$ if we can derive χ from the assumption φ as well as from the assumption ψ. Formally the rule is written as:

$$\frac{\varphi \lor \psi \qquad \boxed{\begin{array}{l}\varphi \ \text{assumption}\\ \vdots \\ \chi\end{array}} \qquad \boxed{\begin{array}{l}\psi \ \text{assumption}\\ \vdots \\ \chi\end{array}}}{\chi} \lor_e$$

Exercise 4.18

Give the proof for the sequent $p \lor q \;\vdash\; q \lor p$.

Solution.

$p \lor q \;\vdash\; q \lor p$

1.	$p \lor q$	premise
2.	p	assumption
3.	$q \lor p$	$\lor_{i_2} 2$
4.	q	assumption
5.	$q \lor p$	$\lor_{i_1} 4$
6.	$q \lor p$	$\lor_e 1, 2-3, 4-5$

Exercise 4.19

Give the proof for the sequent $q \to r \;\vdash\; (p \lor q) \to (p \lor r)$.

Solution.

1.	$q \to r$	premise
2.	$p \lor q$	assumption
3.	p	assumption
4.	$p \lor r$	$\lor_{i_1} 2$
5.	q	assumption
6.	r	$\to_e 5, 1$
7.	$p \lor r$	$\lor_{i_6}$
8.	$p \lor r$	$\lor_e 2, 3-4, 5-7$
9.	$p \lor q \to (p \lor r)$	$\to_i 2-8$

Exercise 4.20

Give the proof for the sequent: $p \wedge (q \vee r), p, (q \vee r) \ \vdash \ (p \wedge q) \vee (p \wedge r)$.

Solution.

1.	$p \wedge (q \vee r)$	premise
2.	p	premise
3.	$(q \vee r)$	premise
4.	q	assumption
5.	$p \wedge q$	$\wedge$i 2,4
6.	$(p \wedge q) \vee (p \wedge r)$	$\vee_{i_1} 5$
7.	r	assumption
8.	$p \wedge r$	$\wedge$i 2,7
9.	$(p \wedge q) \vee (p \wedge r)$	$\vee_{i_2} 8$
10.	$(p \wedge q) \vee (p \wedge r)$	$\vee_e 3, 4-6, 7-9$

The 'Copy'-Rule

The copy rules allows us to repeat any formula that we have already proven. This is helpful when we need to conclude a box with a formula that we have already proven outside of the box. In this case, the formula can simply be copied into the box which can then be closed.

Exercise 4.21

Give the proof for the sequent $p \ \vdash \ q \rightarrow (p \vee t)$.

Solution.

1.	p	premise
2.	q	assumption
3.	p	copy 1
4.	$p \vee t$	$\vee_{i_1} 3$
5.	$q \rightarrow (p \vee t)$	$\rightarrow_i 2-4$

Definition 4.1 (Theorem) A formula φ in propositional logic with a valid sequent $\vdash \ \varphi$ is called a *theorem*.

Exercise 4.22

Give the proof for the sequent $\vdash \; p \to q \to p$.

Solution.

1.	p	assumption
2.	q	assumption
3.	p	copy 1
4.	$q \to p$	$\to_i$ 2 − 3
5.	$p \to q \to p$	$\to_i$ 1 − 4

The Contradiction-Elimination Rule

Definition 4.2 (Contradiction) A *contradiction* is an expressions of the form $\varphi \wedge \neg\varphi$ or $\neg\varphi \wedge \varphi$, where φ is any propositional formula.

Any formula can be derived from a contradiction. Therefore, the proof rule for contradiction elimination looks as follows.

$$\frac{\bot}{\varphi} \; \bot_e$$

The rule expresses that we can derive anything from a contradiction. Lets say, that our two premises say "Sunflowers are plants" and "Sunflowers are not plants". These two premises cannot be true at the same time, and we can infer a contradiction. From the contradiction we can infer anything, like e.g., *Therefore*, "drinking energy drinks helps you sleep better." If a formula on the left hand side of an entailment relation is false, the entire sequent is trivially true.

The Negation-Elimination Rule

We use the negation-elimination rule to derive a contradiction from the given formulas φ and $\neg\varphi$. Formally the rule is written as:

$$\frac{\varphi \qquad \neg\varphi}{\bot} \; \neg_e$$

Exercise 4.23

Give the proof for the sequent $\neg p, p \vdash q$.

Solution.

1.	$\neg p$	premise
2.	p	premise
3.	$\bot$	$\neg_e 2, 1$
4.	q	$\bot_e 3$

Exercise 4.24

Give the proof for the sequent $p \vee \neg q \vdash q \rightarrow (p \vee r)$.

Solution.

1.	$p \vee \neg q$	premise
2.	q	assumption
3.	p	assumption
4.	$p \vee r$	$\vee_{i_1} 3$
5.	$\neg q$	assumption
6.	$\bot$	$\neg_e 2, 5$
7.	$p \vee r$	$\bot_e 6$
8.	$p \vee r$	$\vee_e 1, 3-4, 5-7$
9.	$q \rightarrow (p \vee r)$	$\rightarrow_i 2-8$

The Negation-Introduction Rule

Lets assume that we make an assumption which gets us a contradiction. If this is the case, our assumption must be false. The ¬i rule captures this intuition:

$$\frac{\boxed{\begin{array}{l}\varphi \text{ assumption} \\ \vdots \\ \bot\end{array}}}{\neg\varphi}\ \neg_i$$

Exercise 4.25

Give the proof for the sequent $p \to \neg q, q \vdash \neg p$.

Solution.

1.	$p \to \neg q$	premise
2.	q	premise
3.	p	assumption
4.	$\neg q$	$\to_e$ 3, 1
5.	$\bot$	$\neg_e 2, 4$
6.	$\neg p$	$\neg_i 3 - 5$

Exercise 4.26

Give the proof for the sequent $p \to \neg p \vdash \neg p$.

Solution.

1.	$p \to \neg p$	premise
2.	p	assumption
3.	$\neg p$	$\to_e$ 1, 2
4.	$\bot$	$\neg_e 2, 3$
5.	$\neg p$	$\neg_i 2 - 4$

Exercise 4.27

Give the proof for the sequent $p \wedge \neg q \to r, \neg r, p \vdash q$.

Solution.

1.	$p \wedge \neg q \to r$	premise
2.	$\neg r$	premise
3.	p	premise
4.	$\neg q$	assumption
5.	$p \wedge \neg q$	$\wedge_i 3, 4$
6.	r	$\to_e$ 1, 5
7.	$\bot$	$\neg_e 6, 2$
8.	$\neg\neg q$	$\neg_i 2 - 4$
9.	q	$\neg\neg_e 8$

The Proof-by-Contradiction Rule (PBC)

Another handy derived-rule is called the proof-by-contradiction rule (PBC). It is very similar to the $\neg_i$ rule. The rule states that if from $\neg\varphi$ we obtain a contradiction, then we are allowed to conclude φ:

$$\frac{\boxed{\begin{array}{ll}\neg\varphi & \text{assumption}\\ \vdots & \\ \bot & \end{array}}}{\varphi}\ \text{PBC}$$

Exercise 4.28

Give the proof for the sequent $\neg p \to \neg q, q \ \vdash \ p$.

Solution.

1.	$\neg p \to \neg q$	premise
2.	q	premise
3.	$\neg p$	assumption
4.	$\neg q$	$\to_e$ 3, 1
5.	$\bot$	$\neg_e$ 2, 4
6.	p	PBC 3 – 5

(Lines 3–5 are enclosed in a box.)

The Law-of-the-Excluded-Middle Rule (LEM)

The LEM simply says that $\varphi \lor \neg\varphi$ is true. For every formula φ it holds that it is either true or false, therefore the sequent $\vdash \ \varphi \lor \neg\varphi$ is valid.

$$\frac{}{\varphi \lor \neg\varphi}\ \text{LEM}$$

Exercise 4.29

Give the proof for the sequent $p \vdash q \wedge r \vee \neg(q \wedge r)$.

Solution.

1.	p	premise
2.	$q \wedge r \vee \neg(q \wedge r)$	LEM

Exercise 4.30

Give the proof for the sequent $p \to q \vdash \neg p \vee q$ using LEM.

Solution.

1.	$p \to q$	premise
2.	$\neg p \vee p$	LEM
3.	$\neg p$	assumption
4.	$\neg p \vee q$	$\vee_{i_1}$ 3
5.	p	assumption
6.	q	$\to_e$ 1,5
7.	$\neg p \vee q$	$\vee_{i_2}$ 6
8.	$\neg p \vee q \vee_e 2, 3 - 5, 5 - 7$	

Tips for making your own Natural Deduction Proof

- **Start a proof.** At the top of your page write the premises, at the bottom write the conclusion.
- **Work in both directions to fill the gap.** Work from the top to the bottom by working with the premises, and simultaneously work upwards by using the conclusion.
- **Look first at the conclusion.** If the conclusion is of the form $\varphi \to \psi$, then immediately apply $\to_i$. You still have to fill the gap in the box, but you have an extra assumption to work with and a simpler conclusion you try to reach. Similar, if your conclusion is of the form $\neg\varphi$, apply $\neg_i$ to make your life easier.
- **Assumption boxes.** At any time you can introduce a formula as assumption, by choosing a proof rule that opens the box. The box defines the scope of the assumption. By opening a box you introduce an assumption. But don't forget, you have to close the box precisely as defined by the applied proof rule.

- **What rule should you apply?** The rules $\rightarrow_i$ and $\neg_i$ make your life easier, apply them whenever you can. There is no easy recipe for when to use the other rules. The best way to get the hang of it is doing many proofs by yourself.

4.2 Soundness and Completeness of ND

Soundness

Natural deduction for propositional logic is sound. Therefore, any sequent that can be proven is a correct semantic entailment.

$$\varphi_1, \varphi_2, ..., \varphi_n \vdash \psi \qquad \Rightarrow \qquad \varphi_1, \varphi_2, ..., \varphi_n \vDash \psi$$

So, if we have proven with natural deduction that a sequent $\varphi_1, \varphi_2, ..., \varphi_n$ is valid, then for all valuations in which all premises $\varphi_1, \varphi_2, ..., \varphi_n$ evaluate to *true*, ψ evaluates to *true* as well.

From soundness also follows that *if the semantic entailment relation does not hold, the sequent cannot be proven using natural deduction.*

$$\varphi_1, \varphi_2, ..., \varphi_n \nvDash \psi \qquad \Rightarrow \qquad \varphi_1, \varphi_2, ..., \varphi_n \nvdash \psi$$

Completeness

Natural deduction for propositional logic is complete. Therefore, any sequent that is a correct semantic entailment can be proven.

$$\varphi_1, \varphi_2, ..., \varphi_n \vDash \psi \qquad \Rightarrow \qquad \varphi_1, \varphi_2, ..., \varphi_n \vdash \psi$$

From completeness also follows that *if a sequent is not provable that means it is no correct semantic entailment.*

$$\varphi_1, \varphi_2, ..., \varphi_n \nvdash \psi \qquad \Rightarrow \qquad \varphi_1, \varphi_2, ..., \varphi_n \nvDash \psi$$

Corollary: Soundness and Completeness

Natural deduction for propositional logic is sound and complete.

Let $\varphi_1, \varphi_2, ..., \varphi_n, \psi$ be formulas of propositional logic. Then $\varphi_1, \varphi_2, ..., \varphi_n \vDash \psi$ is holds if and only if the sequent $\varphi_1, \varphi_2, ..., \varphi_n \vdash \psi$ is valid.

4.2.1 Invalid Sequents

To show that a sequent is invalid, we need to find a *counter example.* A counter example is a model, that *satisfies all premises but falsifies the conclusion.*

Exercise 4.31

Show that the sequent $p \wedge q \quad \vdash \quad \neg p$ is *not* valid by finding a counter-example.
The model

$$\mathcal{M} : p = T, q = T$$

is a counter example, since it satisfies the premise, i.e., $\mathcal{M} \models p \wedge q$, and it does not satisfy the conclusion, i.e., $\mathcal{M} \not\models \neg p$.

Exercise 4.32

Find two counter-examples for the sequent $p \vee q \quad \vdash \quad p \wedge q$.

Solution.

$$\begin{array}{ll} \mathcal{M} : p = T,\ q = F & \mathcal{M} : p = F,\ q = T \\ \mathcal{M} \models p \vee q,\ \mathcal{M} \not\models p \wedge q & \mathcal{M} \models p \vee q,\ \mathcal{M} \not\models p \wedge q \\ \textit{Therefore, } p \vee q \nvdash p \wedge q & \textit{Therefore, } p \vee q \nvdash p \wedge q \end{array}$$

List of Definitions