#### Logic and Computability



S C I E N C E P A S S I O N T E C H N O L O G Y

# **Combinational Equivalence Checking**

MY HOBBY: EMBEDDING NP-COMPLETE PROBLEMS IN RESTAURANT ORDERS

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https://xkcd.com/287/

CHOTCHKIES R	ESTAURANT
- APPETIZER	s
MIXED FRUIT	2.15
FRENCH FRIES	2.75
SIDE SALAD	3.35
HOT WINGS	3.55
MOZZARELLA STICKS	4.20
SAMPLER PLATE	5.80
- SANDWICHES	$\sim$
BARBECUE	6 55

WED LIKE EXACTLY \$ 15.05 WORTH OF APPETIZERS, PLEASE. ... EXACTLY? UHH ... HERE, THESE PAPERS ON THE KNAPSACK PROBLEM MIGHT HELP YOU OUT. LISTEN. I HAVE SIX OTHER TABLES TO GET TO -- AS FAST AS POSSIBLE, OF COURSE. WANT SOMETHING ON TRAVELING SALESMAN?

# Recap - Topics we discussed so far

- Propositional Logic
  - Syntax and Semantics
- SAT Solving (DPLL)
  - (Efficiently) solve huge formulas
- BDDs
  - Data structure to efficiently store and manipulate formulas
- Natural Deduction
  - Prove that arguments in prop. logic are valid

# Plan of Today

#### First Part – A few Basic Concepts of Propositional Logic

- Last lecture about propositional logic
  - Next week: predicate logic
- Several basic concepts
  - Relations between Satisfiability, Validity, and Equivalence
  - Normal Forms: CNF, DNF
  - Logical equivalences: Distributive laws, De Morgan's law...
- Tseitin Encoding
  - Computes equisatisfiable formula in CNF
- Equivalence checking via reduction to SAT

#### Second Part – Z3

- Introduction to SMT solver Z3
- Focus on solving formulas in propositional logic



- Algorithm Decide equivalence of combinational circuits
   Based on reduction to Satisfiability
- Relations between Satisfiability, Validity, and Equivalence
- Normal Forms
- Tseitin Encoding
  - Algorithm to translate formula in equisatisfiable formula in CNF



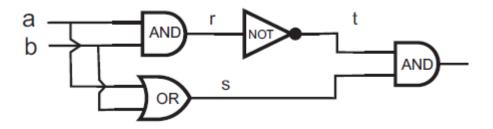
### Learning Outcomes

After this lecture...

- 1. students can apply the algorithm to check for equivalence based on the reduction to SAT.
- 2. students can explain the relation between satisfiability, validity, and equivalence.
- 3. students can rewrite and simplify formulas by applying logical equivalences.
- 4. students can construct the CNF and DNF normal forms of formulas via truth tables.
- 5. students can apply Tseitin's algorithm to construct formulas in CNF.
- 6. students can explain the concept of equisatisfiability.

# **Combinational Equivalence Checking**

- Circuit Optimization and Synthesis Tools
  - Big Market
  - Tools can make mistakes!
  - Need to check for equivalence

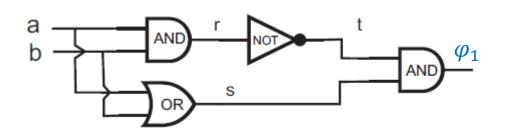




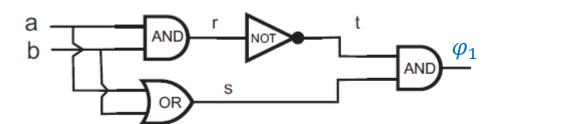
# Algorithm - Circuit Equivalence via Truth Tables

- Using Truth Tables: Check for  $\phi \models \psi$  and  $\psi \models \phi$ ?
  - i.e.,  $\phi$  and  $\psi$  are true for the same models
  - Exponentially large
  - $\rightarrow$  Not practicable!
- Better way: Reduction to SAT

**Step 1: Encode** C<sub>1</sub> and C<sub>2</sub> into formulas:







**Step 1: Encode** C<sub>1</sub> **and** C<sub>2</sub> **into formulas:** 

$$\begin{aligned} \varphi_1 &= t \wedge s \\ &= \neg r \wedge (a \lor b) \\ &= \neg (a \land b) \wedge (a \lor b) \end{aligned}$$



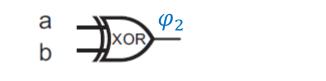
 $\varphi_2 = (\boldsymbol{a} \land \neg \boldsymbol{b}) \lor (\neg \boldsymbol{a} \land \boldsymbol{b})$ 

 $\varphi_1$ 

AND

Step 1: Encode C<sub>1</sub> and C<sub>2</sub> into formulas:





AND

s

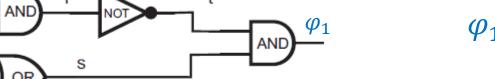
$$\varphi_2 = (\boldsymbol{a} \wedge \neg \boldsymbol{b}) \vee (\neg \boldsymbol{a} \wedge \boldsymbol{b})$$

Circuits are **equivalent**  $\Leftrightarrow \varphi_1 \oplus \varphi_2$  is **unsatisfiable**.

а

b

Step 1: Encode C<sub>1</sub> and C<sub>2</sub> into formulas:



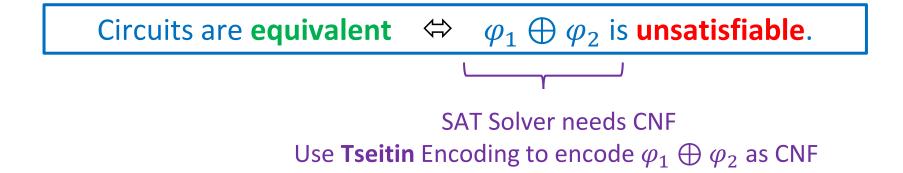
$$\varphi_1 = \neg(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge (\boldsymbol{a} \vee \boldsymbol{b})$$

a b  $\varphi_2$ 

а

b

$$\varphi_2 = (\boldsymbol{a} \wedge \neg \boldsymbol{b}) \vee (\neg \boldsymbol{a} \wedge \boldsymbol{b})$$



- 1. Encode  $C_1$  and  $C_2$  into two formulas  $\varphi_1$  and  $\varphi_2$
- 2. Compute the Conjunctive Normal Form (CNF) of  $\varphi_1 \oplus \varphi_2$ 
  - Use Tseitin Encoding
- 3. Give  $CNF(\varphi_1 \oplus \varphi_2)$  to a **SAT solver**
- 4.  $C_1$  and  $C_2$  are **equivalent** if and only if  $\varphi_1 \oplus \varphi_2$  is **UNSAT**



- Algorithm Decide equivalence of combinational circuits
   Based on reduction to Satisfiability
- Relations between Satisfiability, Validity, and Equivalence
- Normal Forms
- Tseitin Encoding





# **Duality: Validity and Satisfiability**

- $\phi$  is valid  $\Leftrightarrow \neg \phi$  is not satisfiable  $\phi$  is satisfiable  $\Leftrightarrow \neg \phi$  is not valid
- Example:
  - $\phi = (x \lor \neg x)$  is valid. Truth Table: All rows **T**.
  - $\neg \phi = \neg (x \lor \neg x) \equiv \neg x \land x$  is not satisfiable. Truth Table: All rows **F**.
- Only one decision procedure needed

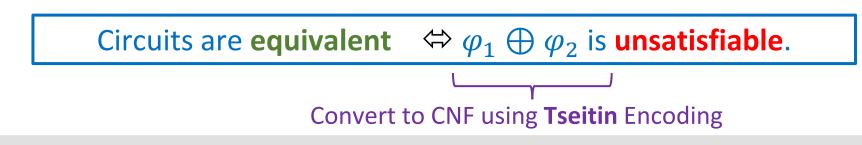
#### Reductions

Only one decision procedure needed

Solve using	$\phi$ satisfiable?	$\phi$ valid?	$\phi\equiv\psi$ ?
Satisfiability	$\checkmark$	<i>¬∲</i> not satisfiable?	$\phi \oplus \psi$ not satisfiable?
Validity	$\neg \phi$ not valid?		$\phi \leftrightarrow \psi$ valid?
Equivalence	$\phi ot\equiv \perp$ ?	$\phi \equiv  op ?$	



- Algorithm Decide equivalence of combinational circuits
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#### **Normal Forms**

- Literal: propositional variable or its negation
  - Example: p,  $\neg q$
- Disjunctive Normal Form (DNF)
  - Disjunction of conjunction of literals:

 $(a_1 \wedge a_2 \wedge \cdots \wedge a_n) \vee (b_1 \wedge \cdots \wedge b_m) \vee \cdots$ 

where each  $a_i$ ,  $b_j$  is a literal

- Conjunctive Normal Form (CNF)
  - Conjunction of disjunctions of literals:

$$(a_1 \lor a_2 \lor \cdots \lor a_n) \land (b_1 \lor \cdots \lor b_m) \land \cdots$$

where each  $a_i$ ,  $b_j$  is a literal

#### Ways to Obtain a CNF

- SAT Solvers require formula in CNF as input
- Obtain CNF via Truth Table
  - Exponential size
- Obtain CNF via logical equivalences (De Morgan's laws, Distributive laws...)
   Exponential size
- Tseitin Encoding
  - Use auxiliary variables
  - Linear blow-up
  - Produces equisatisfiable formula with linear blowup

#### **DNF from Truth Table**

S

Example:

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p	q	r	$(r \lor q) \to (p \land \neg q)$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

#### Example:

p	q	r	$(r \lor q)  ightarrow (p \land \neg q)$		
0	0	0	1		$\neg r$
0	0	1	0	_	
0	1	0	0		
0	1	1	0	_	
1	0	0	1	(	1
1	0	1	1	<b></b>	
1	1	0	0		
1	1	1	0		
_					

Enumerate satisfying models, connect satisfying models with disjunctions.

 $\neg p \land \neg q \land \neg r$ 

 $p \land \neg q \land \neg r$  $p \land \neg q \land r$ 

DNF:  $(\neg p \land \neg q \land \neg r) \lor (p \land \neg q \land \neg r) \lor (p \land \neg q \land r)$ 

#### **CNF from Truth Table**

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#### Example:

21

p	q	r	$(p \lor \neg q)  ightarrow r$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

### **CNF from Truth Table**

Exa	amp	ole:			connect with conju
p	q	r	$(p \lor \neg q)  ightarrow r$		
0	0	0	0		$p \lor q \lor r$
0	0	1	1		
0	1	0	1		
0	1	1	1		
1	0	0	0		$\neg p \lor q \lor r$
1	0	1	1		
1	1	0	0	(	$\neg p \lor \neg q \lor r$
1	1	1	1		• •

**Exclude falsifying models** by requiring that at least one literal per falsifying model must be different, connect with conjunctions.

 $\mathsf{CNF}: (p \lor q \lor r) \land (\neg p \lor q \lor r) \land (\neg p \lor \neg q \lor r)$ 



- Algorithm Decide equivalence of combinational circuits
   Based on reduction to Satisfiability
- Relations between Satisfiability, Validity, and Equivalence
- Normal Forms
- Tseitin Encoding





### **Tseitin Encoding**

- Produces equisatisfiable formula in CNF with linear blowup
- Trick: Use auxiliary variables
- Definition of equisatisfiability:

 $\phi$  and  $\psi$  are **equisatisfiable**  $\iff$  either both are satisfiable, or both are unsatisfiable

For equivalence checking, we only need the info SAT or UNSAT

# **Tseitin Encoding**

- Step 1
  - Assign new variables to each sub-formula
- Step 2
  - Add explanation for each new variable
- Step 3
  - Apply Tseitin Rewrite Rules to obtain equisatisfiable CNF

$$\begin{split} \chi \leftrightarrow (\varphi \lor \psi) & \Leftrightarrow \quad (\neg \varphi \lor \chi) \land (\neg \psi \lor \chi) \land (\neg \chi \lor \varphi \lor \psi) \\ \chi \leftrightarrow (\varphi \land \psi) & \Leftrightarrow \quad (\neg \chi \lor \varphi) \land (\neg \chi \lor \psi) \land (\neg \varphi \lor \neg \psi \lor \chi) \\ \chi \leftrightarrow \neg \varphi & \Leftrightarrow \quad (\neg \chi \lor \neg \varphi) \land (\varphi \lor \chi) \end{split}$$

#### <sup>26</sup> Example – Tseitin Encoding

Use Tseitin encoding to compute the CNF of  $\varphi = ((p \lor q) \land r) \lor \neg p$ .

#### **Rewrite Rules**

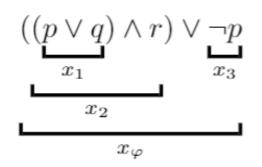
$\chi \leftrightarrow (\varphi \lor \psi)$	$\Leftrightarrow$	$(\neg \varphi \lor \chi) \land (\neg \psi \lor \chi) \land (\neg \chi \lor \varphi \lor \psi)$
$\chi \leftrightarrow (\varphi \wedge \psi)$	$\Leftrightarrow$	$(\neg \chi \lor \varphi) \land (\neg \chi \lor \psi) \land (\neg \varphi \lor \neg \psi \lor \chi)$
$\chi\leftrightarrow\neg\varphi$	$\Leftrightarrow$	$(\neg \chi \lor \neg \varphi) \land (\varphi \lor \chi)$

#### <sup>27</sup> Example – Tseitin Encoding

Use Tseitin encoding to compute the CNF of  $\varphi = ((p \lor q) \land r) \lor \neg p$ .

**Rewrite Rules** 

$\chi \leftrightarrow (\varphi \lor \psi)$	$\Leftrightarrow$	$(\neg \varphi \lor \chi) \land (\neg \psi \lor \chi) \land (\neg \chi \lor \varphi \lor \psi)$
$\chi \leftrightarrow (\varphi \wedge \psi)$	$\Leftrightarrow$	$(\neg \chi \lor \varphi) \land (\neg \chi \lor \psi) \land (\neg \varphi \lor \neg \psi \lor \chi)$
$\chi\leftrightarrow\neg\varphi$	$\Leftrightarrow$	$(\neg \chi \lor \neg \varphi) \land (\varphi \lor \chi)$



$$CNF(\varphi) = (\neg p \lor x_1) \land (\neg q \lor x_1) \land (\neg x_1 \lor p \lor q)$$
  
 
$$\land (\neg x_2 \lor x_1) \land (\neg x_2 \lor r) \land (\neg x_1 \lor \neg r \lor x_2)$$
  
 
$$\land (\neg x_3 \lor \neg p) \land (p \lor x_3)$$
  
 
$$\land (\neg x_2 \lor x_{\varphi}) \land (\neg x_3 \lor x_{\varphi}) \land (\neg x_{\varphi} \lor x_2 \lor x_3)$$
  
 
$$\land x_{\varphi}$$

#### <sup>28</sup> Example – Tseitin Encoding

Use Tseitin encoding to compute the CNF of  $\varphi = \neg(a \lor \neg b) \lor (\neg a \land c)$ .

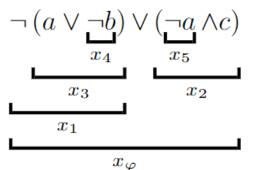


#### **Rewrite Rules**

$\chi \leftrightarrow (\varphi \lor \psi)$	$\Leftrightarrow$	$(\neg \varphi \lor \chi) \land (\neg \psi \lor \chi) \land (\neg \chi \lor \varphi \lor \psi)$
$\chi \leftrightarrow (\varphi \wedge \psi)$	$\Leftrightarrow$	$(\neg \chi \lor \varphi) \land (\neg \chi \lor \psi) \land (\neg \varphi \lor \neg \psi \lor \chi)$
$\chi\leftrightarrow\neg\varphi$	$\Leftrightarrow$	$(\neg \chi \lor \neg \varphi) \land (\varphi \lor \chi)$

#### <sup>29</sup> Example – Tseitin Encoding

Use Tseitin encoding to compute the CNF of  $\varphi = \neg(a \lor \neg b) \lor (\neg a \land c)$ .



 $\chi \leftrightarrow (\varphi \lor \psi) \quad \Leftrightarrow \quad (\neg \varphi \lor \chi) \land (\neg \psi \lor \chi) \land (\neg \chi \lor \varphi \lor \psi)$  $\chi \leftrightarrow (\varphi \land \psi) \quad \Leftrightarrow \quad (\neg \chi \lor \varphi) \land (\neg \chi \lor \psi) \land (\neg \varphi \lor \neg \psi \lor \chi)$  $\chi \leftrightarrow \neg \varphi \quad \Leftrightarrow \quad (\neg \chi \lor \neg \varphi) \land (\varphi \lor \chi)$ 

$$CNF(\varphi) = x_{\varphi} \land (\neg x_1 \lor x_{\varphi}) \land (\neg x_2 \lor x_{\varphi}) \land (\neg x_{\varphi} \lor x_1 \lor x_2) \land (\neg x_1 \lor \neg x_3) \land (x_1 \lor x_3) \land (\neg x_1 \lor \neg x_3) \land (\neg x_4 \lor x_3) \land (\neg x_3 \lor a \lor x_4) \land (\neg x_2 \lor x_3) \land (\neg x_2 \lor c) \land (\neg x_5 \lor \neg c \lor x_2) \land (\neg x_4 \lor \neg b) \land (x_4 \lor b) \land (\neg x_5 \lor \neg a) \land (x_5 \lor a)$$

#### <sup>30</sup> Derive Rewrite Rules

•  $\mathbf{r} \leftrightarrow (\mathbf{p} \land \mathbf{q})$  ... rewrite it to a CNF



De-Morgan

 $\neg (a \land b) \equiv \neg a \lor \neg b$  $\neg (a \lor b) \equiv \neg a \land \neg b$ 

Distributive Law

#### 31 **Derive Rewrite Rules**

- $\mathbf{r} \leftrightarrow (\mathbf{p} \land \mathbf{q})$  ... rewrite it to a CNF
- $(r \rightarrow p \land q) \land (p \land q \rightarrow r)$
- $(\neg r \lor (p \land q)) \land (\neg (p \land q) \lor r)$
- $(\neg r \lor p) \land (\neg r \lor q) \land (\neg p \lor \neg q \lor r)$



De-Morgan

 $\neg (a \land b) \equiv \neg a \lor \neg b$  $\neg(a \lor b) \equiv \neg a \land \neg b$ 

Distributive Law

#### <sup>32</sup> Derive Rewrite Rules

•  $r \leftrightarrow (p \lor q)$  ... rewrite it to a CNF



De-Morgan

 $\neg (a \land b) \equiv \neg a \lor \neg b$  $\neg (a \lor b) \equiv \neg a \land \neg b$ 

Distributive Law

#### Derive Rewrite Rules

- $r \leftrightarrow (p \lor q)$  ... rewrite it to a CNF
- $((p \lor q) \rightarrow r) \land (r \rightarrow p \lor q)$
- $(\neg(p \lor q) \lor r) \land (\neg r \lor p \lor q)$
- $((\neg p \land \neg q) \lor r) \land (\neg r \lor p \lor q)$
- $(\neg p \lor r) \land (\neg q \lor r) \land (\neg r \lor p \lor q)$



De-Morgan

 $\neg (a \land b) \equiv \neg a \lor \neg b$  $\neg (a \lor b) \equiv \neg a \land \neg b$ 

**Distributive Law** 



Derive the rewrite rule for  $x \leftrightarrow (p \rightarrow q)$ .

 $a \to b \equiv \neg a \lor b$ De-Morgan

 $\neg (a \land b) \equiv \neg a \lor \neg b$  $\neg (a \lor b) \equiv \neg a \land \neg b$ 

Distributive Law

### <sup>35</sup> Example

Derive the rewrite rule for  $x \leftrightarrow (p \rightarrow q)$ .

$$\begin{aligned} x \leftrightarrow (p \rightarrow q) \Leftrightarrow x \leftrightarrow (p \rightarrow q) \\ \Leftrightarrow (x \rightarrow (p \rightarrow q)) \land ((p \rightarrow q) \rightarrow x) \\ \Leftrightarrow (x \rightarrow (\neg p \lor q)) \land ((\neg p \lor q) \rightarrow x) \\ \Leftrightarrow (\neg x \lor (\neg p \lor q)) \land (\neg (\neg p \lor q) \lor x) \\ \Leftrightarrow (\neg x \lor \neg p \lor q) \land ((\neg \neg p \land \neg q) \lor x) \\ \Leftrightarrow (\neg x \lor \neg p \lor q) \land ((p \land \neg q) \lor x) \\ \Leftrightarrow (\neg x \lor \neg p \lor q) \land ((p \lor x) \land (\neg q \lor x)) \\ \Leftrightarrow (\neg x \lor \neg p \lor q) \land (p \lor x) \land (\neg q \lor x) \end{aligned}$$

 $a \rightarrow b \equiv \neg a \lor b$ De-Morgan $\neg (a \land b) \equiv \neg a \lor \neg b$ 

 $\neg(a \lor b) \equiv \neg a \land \neg b$ 

**Distributive Law** 

 $a \lor (b \land c) \equiv (a \lor b) \land (a \lor c)$ 

 $a \land (b \lor c) \equiv (a \land b) \lor (a \land c)$ 

#### <sup>36</sup> CEC Example

Check whether  $\varphi_1 = a \land \neg b$  and  $\varphi_2 = \neg(\neg a \lor b)$  are equivalent using the reduction to SAT.



#### CEC Example

Check whether  $\varphi_1 = a \land \neg b$  and  $\varphi_2 = \neg(\neg a \lor b)$  are equivalent using the reduction to SAT.

Step 1) Build 
$$\varphi = \varphi_1 \bigoplus \varphi_2$$
  
 $\varphi = \varphi_1 \oplus \varphi_2$   
 $= [\varphi_1 \lor \varphi_2] \land \neg [\varphi_1 \land \varphi_2] =$   
 $= [(a \land \neg b) \lor (\neg (\neg a \lor b))] \land \neg [(a \land \neg b) \land (\neg (\neg a \lor b))]$ 

# Step 2) Compute CNF of $\varphi$ via Tseitin $\begin{bmatrix} \left(a \land \neg b\right) \lor \left(\neg (\neg a \lor b)\right) \end{bmatrix} \land \neg \begin{bmatrix} \left(a \land \neg b\right) \lor \left(\neg (\neg a \lor b)\right) \end{bmatrix} \\ \downarrow \qquad x_7 \qquad x_8 \qquad x_7 \qquad x_8 \\ \downarrow \qquad x_7 \qquad x_8 \qquad x_6 \\ \downarrow \qquad x_7 \qquad x_8 \\ \downarrow \qquad x_8 \\ \downarrow \qquad x_7 \qquad x_8 \\ \downarrow \qquad x_8 \\ \downarrow \qquad x_7 \qquad x_8 \\ \downarrow \qquad x_8 \\ \end{matrix}$

$$CNF(\varphi) = x_{\varphi} \land \\ (\neg x_{\varphi} \lor x_{1}) \land (\neg x_{\varphi} \lor x_{2}) \land (\neg x_{1} \lor \neg x_{2} \lor x_{\varphi}) \land \\ (\neg x_{1} \lor \neg x_{2}) \land (x_{1} \lor x_{2}) \land \\ (\neg x_{3} \lor x_{1}) \land (\neg x_{4} \lor x_{1}) \land (\neg x_{1} \lor x_{3} \lor x_{4}) \land \\ (\neg x_{3} \lor a) \land (\neg x_{3} \lor x_{7}) \land (\neg a \lor \neg x_{7} \lor x_{3}) \land \\ (\neg x_{4} \lor \neg x_{6}) \land (x_{4} \lor x_{6}) \land \\ (\neg x_{8} \lor x_{6}) \land (\neg b \lor x_{6}) \land (\neg x_{6} \lor x_{8} \lor b) \land \\ (\neg x_{7} \lor \neg b) \land (x_{7} \lor b) \land \\ (\neg x_{8} \lor \neg a) \land (x_{8} \lor a)$$

Step 3) Check via SAT Solver: Is the CNF of  $\varphi$  satisfiable? Step 4) Interpret result:  $\varphi_1$  and  $\varphi_2$  are **equivalent** if and only if  $\varphi_1 \bigoplus \varphi_2$  is **UNSAT** 

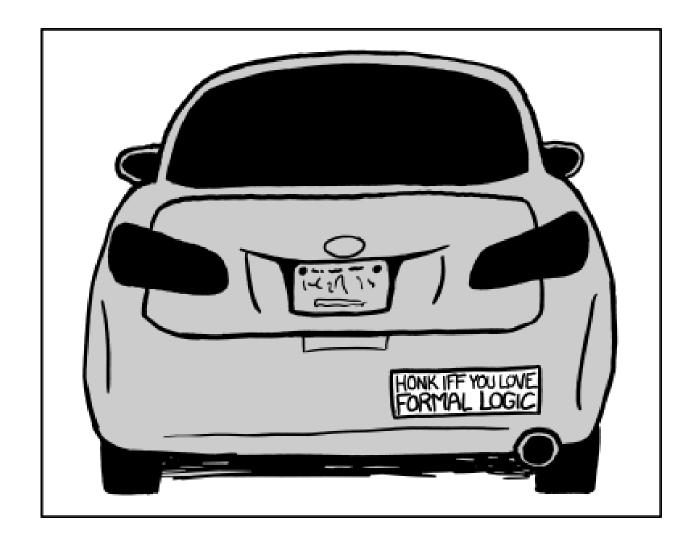
# Learning Outcomes

After this lecture...

- 1. students can apply the algorithm to check for equivalence based on the reduction to SAT.
- 2. students can explain the relation between satisfiability, validity, and equivalence.
- 3. students can rewrite and simplify formulas by applying logical equivalences.
- 4. students can construct the CNF and DNF normal forms of formulas via truth tables.
- 5. students can apply Tseitin's algorithm to construct formulas in CNF.
- 6. students can explain the concept of equisatisfiability.



#### Thank You



https://xkcd.com/1033/