Logic and Computability SS24, Assignment 3

Due: 24. 04. 2024, 23:59

SOLUTION

1 Combinational Equivalence Checking

We list the *Tseitin-rewriting rules* to be applied for the following examples.

$$\begin{aligned}
\chi \leftrightarrow (\varphi \vee \psi) &\quad \Leftrightarrow (\neg\varphi \vee \chi) \wedge (\neg\psi \vee \chi) \wedge (\neg\chi \vee \varphi \vee \psi) \\
\chi \leftrightarrow (\varphi \wedge \psi) &\quad \Leftrightarrow (\neg\chi \vee \varphi) \wedge (\neg\chi \vee \psi) \wedge (\neg\varphi \vee \neg\psi \vee \chi) \\
\chi \leftrightarrow \neg\varphi &\quad \Leftrightarrow (\neg\chi \vee \neg\varphi) \wedge (\chi \vee \varphi)
\end{aligned}$$

1. [3 points] Apply the Tseitin transformation to $\varphi = \neg(\neg b \wedge \neg c) \vee (\neg c \wedge a)$. For each variable you introduce, clearly indicate which subformula it represents.

Solution

$$\underbrace{\underbrace{\neg \underbrace{(\underbrace{\neg b}_{x_4} \wedge \underbrace{\neg c}_{x_5})}_{x_3}}_{x_1} \vee \underbrace{(\underbrace{\neg c}_{x_5} \wedge a)}_{x_2}}_{x_\varphi}$$

$$\begin{aligned}
CNF(\varphi) = x_\varphi \wedge & \\
& (\neg x_1 \vee x_\varphi) \wedge (\neg x_2 \vee x_\varphi) \wedge (\neg x_\varphi \vee x_1 \vee x_2) \wedge \\
& (\neg x_1 \vee \neg x_3) \wedge (x_1 \vee x_3) \wedge \\
& (\neg x_3 \vee x_4) \wedge (\neg x_3 \vee x_5) \wedge (\neg x_4 \vee \neg x_5 \vee x_3) \wedge \\
& (\neg x_2 \vee x_5) \wedge (\neg x_2 \vee a) \wedge (\neg x_5 \vee \neg a \vee x_2) \wedge \\
& (\neg x_4 \vee \neg b) \wedge (x_4 \vee b) \wedge \\
& (\neg x_5 \vee \neg c) \wedge (x_5 \vee c)
\end{aligned}$$

2. [3 points] Apply the Tseitin transformation to $\varphi = \neg(p \to q) \land (r \land p)$. For each variable you introduce, clearly indicate which subformula it represents. Derive the Tseitin transformation rule for $\to$ or transform the input such that you can use the rules above.

Solution

$$\underbrace{\underbrace{\neg \underbrace{(p \to q)}_{x_3}}_{x_1} \land \underbrace{(r \land p)}_{x_2}}_{x_\varphi}$$

We are going to derive the Tseitin transformation rule for an implication:

$$\begin{aligned}
x \leftrightarrow (p \to q) &\Leftrightarrow x \leftrightarrow (p \to q) \\
&\Leftrightarrow (x \to (p \to q)) \land ((p \to q) \to x) \\
&\Leftrightarrow (x \to (\neg p \lor q)) \land ((\neg p \lor q) \to x) \\
&\Leftrightarrow (\neg x \lor (\neg p \lor q)) \land (\neg(\neg p \lor q) \lor x) \\
&\Leftrightarrow (\neg x \lor \neg p \lor q) \land ((\neg\neg p \land \neg q) \lor x) \\
&\Leftrightarrow (\neg x \lor \neg p \lor q) \land ((p \land \neg q) \lor x) \\
&\Leftrightarrow (\neg x \lor \neg p \lor q) \land ((p \lor x) \land (\neg q \lor x)) \\
&\Leftrightarrow (\neg x \lor \neg p \lor q) \land (p \lor x) \land (\neg q \lor x)
\end{aligned}$$

$$\begin{aligned}
CNF(\varphi) = x_\varphi \land \\
& (\neg x_\varphi \lor x_1) \land (\neg x_\varphi \lor x_2) \land (\neg x_1 \lor \neg x_2 \lor x_\varphi) \land \\
& (\neg x_1 \lor \neg x_3) \land (x_1 \lor x_3) \land \\
& (\neg x_3 \lor \neg p \lor q) \land (p \lor x_3) \land (\neg q \lor x_3) \land \\
& (\neg x_2 \lor r) \land (\neg x_2 \lor p) \land (\neg r \lor \neg p \lor x_2)
\end{aligned}$$

Solution

$$\underbrace{\underbrace{\neg \underbrace{(\underbrace{\neg p}_{x_4} \vee q)}_{x_3}}_{x_1} \wedge \underbrace{(r \wedge p)}_{x_2}}_{x_\varphi}$$

$$\begin{aligned} CNF(\varphi) = x_\varphi \wedge & \\ & (\neg x_\varphi \vee x_1) \wedge (\neg x_\varphi \vee x_2) \wedge (\neg x_1 \vee \neg x_2 \vee x_\varphi) \wedge \\ & (\neg x_1 \vee \neg x_3) \wedge (x_1 \vee x_3) \wedge \\ & (\neg x_4 \vee x_3) \wedge (\neg q \vee x_3) \wedge (\neg x_3 \vee x_4 \vee q) \wedge \\ & (\neg x_4 \vee \neg p) \wedge (x_4 \vee p) \wedge \\ & (\neg x_2 \vee r) \wedge (\neg x_2 \vee p) \wedge (\neg r \vee \neg p \vee x_2) \end{aligned}$$

2 Predicate Logic

3. [2 points] Translate the following sentence into predicate logic. Be as precise as possible. Give the meaning of any function and predicate symbols you use.

   Every even integer greater than 2 is equal to the sum of two prime numbers.

   Solution

   - $\mathcal{A} = \mathbb{Z}$
   - $E(x)$ ... true if x is even
   - $G(x)$ ... true if x is greater than 2
   - $P(x)$ ... true if x is prime
   - $x = y$ ... true if x is equal to y
   - $x + y$ ... returns the sum of x and y

   $$\forall x \big(E(x) \land G(x) \to \exists a\ \exists b(P(a) \land P(b) \land (x = a + b))\big)$$

4. [2 points] Translate the following sentence into predicate logic. Be as precise as possible. Give the meaning of any function and predicate symbols you use.

   Every person who has the same parents as John Doe and is different from John Doe himself is a sibling of John Doe.

   Solution

   - $\mathcal{A} =$ People
   - $J(x)$... x is John Doe,
   - $S(x, y)$... x and y have the same parents, and
   - $I(x, y)$... x and y are siblings.

   $$\forall y(J(x) \land \neg J(y) \land S(x, y) \to I(x, y))$$

5. [2 points] Given is the following formula in predicate logic

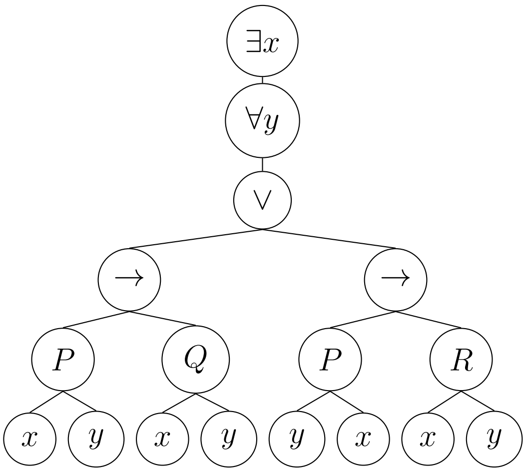
   $$\varphi = \exists x \forall y \Big(\big(P(x, y) \to Q(x, y)\big) \lor \big(P(y, x) \to R(x, y)\big)\Big)$$

   and the model $\mathcal{M}$:

- $\mathcal{A} = \{a, b\}$
- $P^M = \{(a, b), (b, b), (b, a)\}$
- $Q^M = \{(a, a)\}$
- $R^M = \{(b, b)\}$

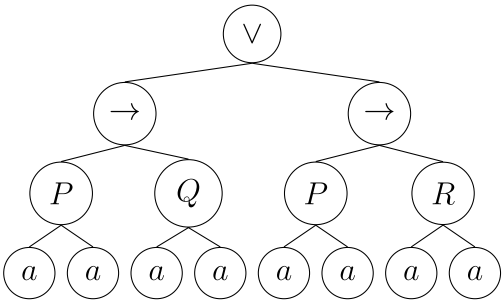
Does the model $\mathcal{M}$ satisfy the formula φ? Evaluate $\mathcal{M}$ using a syntax tree.

Solution



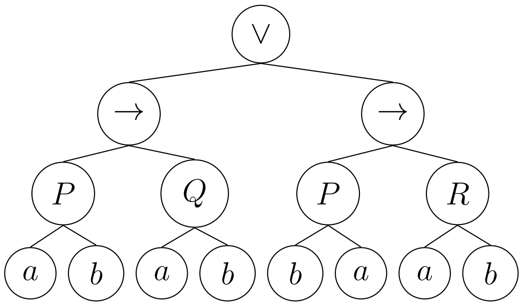
- $x = a$

$x = a \wedge y = a$



$(P(a, a) \rightarrow Q(a, a)) \vee (P(a, a) \rightarrow R(a, a)) =$
$(\bot \rightarrow \top) \vee (\bot \rightarrow \top) = \top$

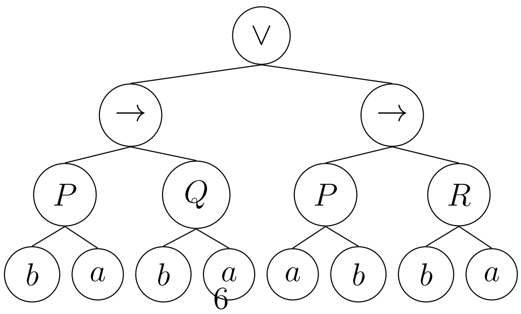
$x = a \wedge y = b$



$(P(a, b) \rightarrow Q(a, b)) \vee (P(b, a) \rightarrow R(b, a)) =$
$(\top \rightarrow \bot) \vee (\top \rightarrow \bot) = \bot$

- $x = b$

$x = b \wedge y = a$



$(P(b, a) \rightarrow Q(b, a)) \vee (P(a, b) \rightarrow R(b, a)) =$
$(\top \rightarrow \bot) \vee (\top \rightarrow \bot) = \bot$

We do not need to evaluate $x = b \wedge y = b$. $M \not\models \varphi$.

6. [3 points] For the formula below, state one model that satisfies the formula, and one model that does not satisfy the formula. Explain your answer by drawing a syntax tree and evaluate your models with the help of this syntax tree.

$$\forall x \exists y (P(f(x), y) \land \neg P(x, f(y)))$$

Solution

There is no solution available for this question yet.