Logic and Computability SS24, Assignment 1

Due: 20. 03. 2024, 23:59

SOLUTION

The DPLL-Algorithm

For the following exercises use the DPLL algorithm (including Boolean Constraint Propagation (BCP), pure literals, and conflict-driven clause learning) to check on paper, if the following CNF formulas are satisfiable.

If the formula is satisfiable, give a satisfying model, else show a complete resolution proof for the formula's unsatisfiability.

- Write down all the steps of the DPLL algorithm,
- draw the conflict graphs,
- and state the resolution proofs for all learned clauses.

Rules:

- When resolving a conflict, only undo the last decision.
- Choose variables for decisions, BCP and pure literals in alphabetical order, starting with the *negative* phase $(\neg a > a > \neg b > b...)$.
- Always try to perform BCP first, before checking for pure literals, before making a decision.
- 1. [3 points] Use the DPLL algorithm with the rules as described above to check whether the following formula in CNF is satisfiable

Clause 1: $\{a, \neg b, c\}$

Clause 2: $\{b, \neg c, d\}$

Clause 3: $\{a, \neg b\}$

Clause 4: $\{a, c\}$

Clause 5: $\{\neg c, \neg d\}$ Clause 6: $\{\neg a, c\}$

Solution							
Step	1	2	3	4	5		
Decision Level	0	1	1	1	1		
Assignment	-	$\neg a$	$\neg a, \neg b$	$\neg a, \neg b, c$	$\neg a, \neg b, c, \\ \neg d$		
Cl. 1: $a, \neg b, c$	$a, \neg b, c$	$\neg b, c$	✓	√	✓		
Cl. 2: $b, \neg c, d$	$b, \neg c, d$	$b, \neg c, d$	$\neg c, d$	d	{} X		
Cl. 3: a, ¬b	$a, \neg b$	$\neg b$	1	✓	✓		
Cl. 4: a, c	a, c	c	c	✓	✓		
Cl. 5: $\neg c, \neg d$	$\neg c, \neg d$	$\neg c, \neg d$	$\neg c, \neg d$		✓		
Cl. 6: $\neg a, c$	$\neg a, c$	✓	✓	✓	✓		
BCP	-	$\neg b$	c	$\neg d$	-		
PL	-	-	-	-	-		
Decision	$\neg a$	-	-	-	-		
Conflict in ste	p 5						
	c $\frac{2}{5}$	d		-)			
	5		5. ¬c'	$\frac{\sqrt{\neg d}}{\neg c \lor a} \qquad 3$	$\frac{.\ a \lor \neg b}{a}$	$4. \ a \lor c$	<u>:</u>
Step	5		5. ¬c		$ \begin{array}{c c} \underline{a} \lor \neg b \\ \hline a \\ \hline 10 \end{array} $	$4. \ a \lor c$	<u>;</u>
	$\begin{array}{c} c \\ \hline \\ 5 \\ \hline \\ 2. \ b \lor \neg c \\ \end{array}$	$\begin{array}{c} -d \\ \hline -d \\ \hline b \lor \neg c \\ \end{array}$		9 0	$\begin{bmatrix} 10 \\ 0 \end{bmatrix}$	4. a∨c	<u>;</u>
Step Decision Level Assignment	$ \begin{array}{c} c \\ 5 \\ 2. \ b \lor \neg c \end{array} $	$\frac{c \vee d}{b \vee \neg c}$	8 0	$\begin{array}{c c} 9 & a \\ \hline 0 & a \\ \end{array}$	10	$4. \ a \lor c$	<u>;</u>
Step Decision Level Assignment Cl. 1: $a, \neg b, c$	$ \begin{array}{c} c \\ \hline c \\ c \\ \hline c \\ \hline c \\ c \\ \hline c \\ c \\ \hline c \\ c \\$	$ \begin{array}{c c} \hline & \neg d \\ \hline & \neg d \\ \hline & \neg d \\ \hline & \bullet \\ \hline & $	8 0 a,c	$ \begin{array}{c c} 9 & \vdots \\ \hline 0 & \\ a, c, \neg d & a, \epsilon \end{array} $	$ \begin{array}{c} a \\ \hline 10 \\ 0 \\ c, \neg d, \\ b \\ \checkmark \end{array} $	$4. \ a \lor c$	<u>.</u>
Step Decision Level Assignment Cl. 1: $a, \neg b, c$ Cl. 2: $b, \neg c, d$	$ \begin{array}{c} c \\ c \\$	$ \begin{array}{c c} \hline & & \\ \hline $	8 0 a, c d b, d	$ \begin{array}{c c} 9 & \vdots \\ 0 & a, c, \neg d \\ \hline $	$\begin{bmatrix} 10 \\ 0 \\ c, \neg d, \\ b \\ \checkmark \end{bmatrix}$	$4. \ a \lor c$	<u> </u>
Step Decision Level Assignment Cl. 1: $a, \neg b, c$ Cl. 2: $b, \neg c, d$ Cl. 3: $a, \neg b$	$ \begin{array}{c} c \\ c \\$	$ \begin{array}{c c} \hline & & \\ \hline $	8 0 a,c 6 b,d	$ \begin{array}{c c} 9 & \vdots \\ \hline 0 & a, c, \neg d \\ \hline \checkmark & b \\ \checkmark & \end{array} $	$ \begin{array}{c c} a \\ \hline 10 \\ \hline 0 \\ \vdots, \neg d, \\ b \\ \checkmark \checkmark \checkmark \end{array} $	$4. a \vee c$	<u>, </u>
Step Decision Level Assignment Cl. 1: $a, \neg b, c$ Cl. 2: $b, \neg c, d$ Cl. 3: $a, \neg b$ Cl. 4: a, c	$ \begin{array}{c} c \\ c \\$	$ \begin{array}{c c} \hline & & \\ \hline $	8 0 a, c d	$ \begin{array}{c c} 9 & \vdots \\ 0 & \\ a, c, \neg d & a, c \end{array} $	$ \begin{array}{c c} a \\ \hline 10 \\ 0 \\ c, \neg d, \\ b \\ \checkmark \\ \checkmark$	4. a ∨ c	<u>;</u>
Step Decision Level Assignment Cl. 1: $a, \neg b, c$ Cl. 2: $b, \neg c, d$ Cl. 3: $a, \neg b$ Cl. 4: a, c Cl. 5: $\neg c, \neg d$	$ \begin{array}{c} c\\ $	$ \begin{array}{c c} \hline & & \\ \hline $	8 0 a, c o	$ \begin{array}{c c} 9 & a \\ \hline 0 & a, c, \neg d \\ \hline $	a 10 0 c, ¬d, b ✓ ✓ ✓ ✓	$4. a \lor c$	<u>;</u>
Step Decision Level Assignment Cl. 1: $a, \neg b, c$ Cl. 2: $b, \neg c, d$ Cl. 3: $a, \neg b$ Cl. 4: a, c Cl. 5: $\neg c, \neg d$ Cl. 6: $\neg a, c$	$ \begin{array}{c} c \\ c \\$	$ \begin{array}{c c} \hline & \neg d \\ \hline & \neg c \\ & \neg c \\ \hline & \neg c \\ & $	8 0 a, c o	$ \begin{array}{c c} 9 & a \\ \hline 0 & a, c, \neg d \\ \hline $	a 10 0 c, ¬d, b ✓ ✓ ✓ ✓ ✓	4. a V c	
Step Decision Level Assignment Cl. 1: $a, \neg b, c$ Cl. 2: $b, \neg c, d$ Cl. 3: $a, \neg b$ Cl. 4: a, c Cl. 5: $\neg c, \neg d$ Cl. 6: $\neg a, c$ Cl. 7: a	$ \begin{array}{c} c \\ c \\$	$ \begin{array}{c c} \hline & & \\ \hline $	8 0 a, c a b, d b, d v v v v v v v v v v v v v v v v v v	$ \begin{array}{c c} 9 & \vdots \\ 0 & a, c, \neg d \\ \hline $	a 10 0 c, ¬d, b ✓ ✓ ✓ ✓	4. a ∨ c	<u>:</u>
Step Decision Level Assignment Cl. 1: $a, \neg b, c$ Cl. 2: $b, \neg c, d$ Cl. 3: $a, \neg b$ Cl. 4: a, c Cl. 5: $\neg c, \neg d$ Cl. 6: $\neg a, c$ Cl. 7: a BCP	$ \begin{array}{c} c \\ c \\$	$ \begin{array}{c c} \hline & & \\ \hline $	8 0 a, c o	9	a 10 0 c, ¬d, b ✓ ✓ ✓ ✓ ✓	$4. a \vee c$	<u>;</u>
Step Decision Level Assignment Cl. 1: $a, \neg b, c$ Cl. 2: $b, \neg c, d$ Cl. 3: $a, \neg b$ Cl. 4: a, c Cl. 5: $\neg c, \neg d$ Cl. 6: $\neg a, c$ Cl. 7: a	$ \begin{array}{c} c \\ c \\$	$ \begin{array}{c c} \hline & & \\ \hline $	8 0 a, c a b, d b, d v v v v v v v v v v v v v v v v v v	9	a 10 0 c, ¬d, b ✓ ✓ ✓ ✓ ✓	$4. a \lor c$	

2. [2 points] Use the DPLL algorithm with the rules as described above to check whether the following formula in CNF is satisfiable

Clause 1: $\{a, b, c\}$

Clause 2: $\{\neg b, \neg c, e\}$

Clause 3: $\{b, e\}$

Clause 4: $\{b, \neg d\}$

Clause 5: $\{\neg c, d\}$

Clause 6: $\{\neg c, e\}$

Clause 7: $\{\neg a, \neg b, \neg c\}$

Clause 8: $\{a, c, \neg e\}$

Solution								
Step	1	2	2	3	4		5]
Decision Level	0	1		2	2		2	_
Assignment	-	-	a	$\neg a, \neg b$	$\neg a, \neg$	$\neg b, c$	$\neg a, \neg b, c, \\ \neg d$	-
Cl. 1: a, b, c	a, b, c	<i>b.</i>	c	\overline{c}	/	,	√	_
Cl. 2: $\neg a, \neg b, \neg c$	$\neg a, \neg b, \neg c$,	√	/		<u> </u>	-
Cl. 3: $a, c, \neg e$	$a, c, \neg e$	c,		$c, \neg e$	/		√	_
Cl. 4: $\neg b, \neg c, e$	$\neg b, \neg c, e$	$\neg b, \neg b, \neg b$		<u>√</u>	/		<u> </u>	-
Cl. 5: b, e	b, e	b,		$\frac{\bullet}{e}$	e		$\frac{}{e}$	-
Cl. 6: $b, \neg d$	$b, \neg d$		$\neg d$	$\frac{c}{\neg d}$	$\neg c$			-
Cl. 7: $\neg c, d$	$rac{\sigma, \cdot d}{\neg c, d}$		$\frac{d}{d}$	$\neg c, d$	$\frac{d}{d}$		{} X	-
Cl. 8: $\neg c, e$								-
BCP	$\neg c, e$	_	e, e	$\neg c, e$	e		e	-
PL	-	-		c	$\neg c$		-	_
	-	-		-	-		-	_
Decision Conflict in step 5	$\neg a$	-	υ	-	-		-	_
	7 →(
	7 →(_7. ¬c	$\frac{d}{d} \times \frac{d}{d} \times \frac{d}$	1. / a V	$\frac{a \lor b \lor}{b}$	<u>' c</u>	6. <i>l</i>	$o \lor \neg d$	
				a	∨ <i>0</i>			
Step	6	7	8		9	10)	
Step Decision Level				a	∨ <i>0</i>	$\frac{10}{2}$	b, d,	
Step Decision Level Assignment	6 1 ¬a	$ \begin{array}{c} 7\\ 1\\ \neg a, b \end{array} $	8	a b, d a	$\begin{array}{c c} & 0 \\ \hline 9 & 2 \\ \hline b, d, \end{array}$	10	$b, d, \neg e$	
Step Decision Level Assignment Cl. 1: a, b, c	6 1	7	$ \begin{array}{c c} 8 \\ \hline 1 \\ \hline \neg a, b \end{array} $	a b, d a	$\begin{array}{c c} & & \\ \hline 9 & & \\ \hline 2 & & \\ \hline b, d, \\ \neg c & & \\ \end{array}$	$ \begin{array}{c} 10 \\ 2 \\ \neg a, b \\ \neg c, \end{array} $	$b, d, \neg e$	
Step Decision Level Assignment Cl. 1: a, b, c Cl. 2: $\neg a, \neg b, \neg c$	6 1 ¬a b, c ✓	7 1 ¬a, b ✓	8 1 ¬a, b	a	$\begin{array}{c c} \checkmark & b \\ \hline 9 \\ \hline 2 \\ \hline b, d, \\ \hline $	$ \begin{array}{c} 10 \\ 2 \\ \neg a, b \\ \neg c, \end{array} $	b, d,	
Step Decision Level Assignment Cl. 1: a, b, c Cl. 2: $\neg a, \neg b, \neg c$ Cl. 3: $a, c, \neg e$	6 1 ¬a b, c ✓ c, ¬e	$ \begin{array}{c} 7 \\ 1 \\ \neg a, b \\ \checkmark \\ \hline c, \neg e \end{array} $	$ \begin{array}{c c} 8 \\ 1 \\ \hline \neg a, b \end{array} $ $ \begin{array}{c c} \checkmark \\ c, \neg \end{array} $	$\begin{array}{c c} a & \\ \hline \\ b, d & \\ \hline \\ e & \\ \hline \end{array}$	9 2 b, d, ¬c ✓	10 2 ¬a, l ¬c, ✓	b, d,	
Step Decision Level Assignment Cl. 1: a, b, c Cl. 2: $\neg a, \neg b, \neg c$ Cl. 3: $a, c, \neg e$ Cl. 4: $\neg b, \neg c, e$	$ \begin{array}{c cccc} 6 & & & \\ 1 & & & \\ & \neg a & & \\ b, c & & \\ \checkmark & & \\ c, \neg e & & \\ & \neg b, \neg c, e & \\ \end{array} $	$ \begin{array}{c c} 7 \\ 1 \\ \neg a, b \\ \hline \checkmark \\ \hline c, \neg e \\ \neg c, e \end{array} $	$ \begin{array}{c c} 8 \\ 1 \\ \hline $	$\begin{array}{c c} a & \\ \hline \\ b, d & \\ \hline \\ e & \\ \end{array}$	9 2 b, d, ¬c ✓	10 2 ¬a, l ¬c, ✓	b, d,	
Step Decision Level Assignment Cl. 1: a, b, c Cl. 2: $\neg a, \neg b, \neg c$ Cl. 3: $a, c, \neg e$ Cl. 4: $\neg b, \neg c, e$ Cl. 5: b, e	$ \begin{array}{c c} 6 \\ 1 \\ \neg a \\ b, c \\ \checkmark \\ c, \neg e \\ \neg b, \neg c, e \\ b, e \end{array} $	$ \begin{array}{c c} 7\\ \hline 1\\ \neg a, b\\ \checkmark\\ \hline \checkmark\\ c, \neg e\\ \neg c, e\\ \checkmark \end{array} $	8 1 ¬a, b ✓ ✓ c, ¬ ¬c, ¬	$\begin{array}{c c} a & \\ \hline \\ b, d & \\ \hline \\ e & \\ \end{array}$	9 2 b, d,	10 2 ¬a, l ¬c, ✓	b, d, ¬e	
Step Decision Level Assignment Cl. 1: a, b, c Cl. 2: $\neg a, \neg b, \neg c$ Cl. 3: $a, c, \neg e$ Cl. 4: $\neg b, \neg c, e$ Cl. 5: b, e Cl. 6: $b, \neg d$	$ \begin{array}{c c} 6 \\ 1 \\ \neg a \\ b, c \\ \checkmark \\ c, \neg e \\ \neg b, \neg c, e \\ b, e \\ b, \neg d \end{array} $	$ \begin{array}{c c} 7 \\ 1 \\ \hline \neg a, b \\ \hline \checkmark \\ \hline c, \neg e \\ \hline \neg c, e \\ \hline \checkmark \end{array} $	8 1 ¬a, b ✓ ✓ ✓ ✓ ✓ ✓ ¬c, ¬ ¬c,	a ¬a, d ¬a, e e • • • • • • • • • • • • • • • • •	9 9 2 b, d, ¬¬c / / / / / / / / / / / / / / / / / /	10 2 ¬a, l ¬c, ✓	b, d, ¬e	
Step Decision Level Assignment Cl. 1: a, b, c Cl. 2: $\neg a, \neg b, \neg c$ Cl. 3: $a, c, \neg e$ Cl. 4: $\neg b, \neg c, e$ Cl. 5: b, e Cl. 6: $b, \neg d$ Cl. 7: $\neg c, d$	$\begin{array}{c c} 6 & \\ 1 & \\ \neg a & \\ b, c & \\ \checkmark & \\ c, \neg e & \\ \neg b, \neg c, e & \\ b, e & \\ b, \neg d & \\ \neg c, d & \\ \end{array}$	7 1 ¬a, b ✓ ✓ c, ¬e ¬c, e ✓ ✓	8 1 ¬a, t ✓ ✓ ✓ ✓ ✓ ✓ ¬c, ¬ ✓ ✓ ✓ ✓	a ¬a, d ¬a, e e • • • • • • • • • • • • • • • • •	9 9 1 2 b, d, ¬¬c / / / / / / / / / / / / / / / / / /	10 2 7a, l 7c, 1	b, d, ¬e	
Step Decision Level Assignment Cl. 1: a, b, c Cl. 2: $\neg a, \neg b, \neg c$ Cl. 3: $a, c, \neg e$ Cl. 4: $\neg b, \neg c, e$ Cl. 5: b, e Cl. 6: $b, \neg d$ Cl. 7: $\neg c, d$ Cl. 8: $\neg c, e$	$\begin{array}{c c} 6 & \\ 1 & \\ \neg a & \\ b, c & \\ \checkmark & \\ c, \neg e & \\ \neg b, \neg c, e & \\ b, \neg d & \\ \neg c, d & \\ \neg c, e & \\ \end{array}$	$ 7 $ $ 1 $ $ \neg a, b $ $ \checkmark $ $ \checkmark $ $ c, \neg e $ $ \neg c, e $ $ \checkmark $ $ \checkmark $ $ \neg c, d $ $ \neg c, e $	8 1 ¬a, t ✓ ✓ ✓ ✓ ✓ ¬c, ¬ ✓ ✓ ✓ ✓ ✓ ¬c, ¬	a ¬a, d ¬a, e e e e e e e e e e e e e e e e e e e	9 9 2 b, d, ¬¬c // // // // // // // // // // // // //	10 2 ¬a, l ¬c, ✓	b, d, ¬e	
Step Decision Level Assignment Cl. 1: a, b, c Cl. 2: $\neg a, \neg b, \neg c$ Cl. 3: $a, c, \neg e$ Cl. 4: $\neg b, \neg c, e$ Cl. 5: b, e Cl. 6: $b, \neg d$ Cl. 7: $\neg c, d$ Cl. 8: $\neg c, e$ Cl. 9: a, b	$ \begin{array}{c c} 6 \\ 1 \\ $	$ \begin{array}{c c} 7\\ 1\\ \hline \neg a, b\\ \checkmark\\ c, \neg e\\ \hline \neg c, e\\ \checkmark\\ \hline \neg c, d\\ \hline \neg c, e\\ \checkmark\\ \hline \end{array} $	8 1 ¬a, b c, ¬ ¬c, √ √ √ √ √ √ √ √ √ √ √ √ √	a ¬a, d ¬a, e e • • • • • • • • • • • • • • • • •	9 9 2 b, d, ¬¬c / / / / / / / / / / / / / / / / / /	10 2 ¬a, l ¬c, ✓	b, d, ¬e	
Step Decision Level Assignment Cl. 1: a, b, c Cl. 2: $\neg a, \neg b, \neg c$ Cl. 3: $a, c, \neg e$ Cl. 4: $\neg b, \neg c, e$ Cl. 5: b, e Cl. 6: $b, \neg d$ Cl. 7: $\neg c, d$ Cl. 8: $\neg c, e$ Cl. 9: a, b BCP	$\begin{array}{c c} 6 \\ 1 \\ \neg a \\ b, c \\ \checkmark \\ c, \neg e \\ \neg b, \neg c, e \\ b, e \\ b, \neg d \\ \neg c, d \\ \neg c, e \\ b \\ b \end{array}$	$ \begin{array}{c c} 7\\ 1\\ \hline \neg a, b\\ \checkmark\\ \hline c, \neg e\\ \hline \neg c, e\\ \checkmark\\ \hline \neg c, d\\ \hline \neg c, e\\ \checkmark\\ \hline -c, e\\ \checkmark\\ \hline -c, e\\ \checkmark\\ \hline -c, e\\ \checkmark\\ \hline -c, e\\ -c, e\\ \hline -c, e\\ -c, e\\ \hline -c, e\\ $	8 1 ¬a, b	a ¬a, d ¬a, e e • • • • • • • • • • • • • • • • •	9 9 2 b, d, ¬c // // // // // // // // // // // // //	10 2 ¬a, l ¬c, ✓	b, d, ¬e	
	$ \begin{array}{c c} 6 \\ 1 \\ $	$ \begin{array}{c c} 7\\ 1\\ \hline \neg a, b\\ \checkmark\\ c, \neg e\\ \hline \neg c, e\\ \checkmark\\ \hline \neg c, d\\ \hline \neg c, e\\ \checkmark\\ \hline \end{array} $	8 1 ¬a, b c, ¬ ¬c, √ √ √ √ √ √ √ √ √ √ √ √ √	a ¬a, d ¬a, e e • • • • • • • • • • • • • • • • •	9 9 2 b, d, ¬¬c / / / / / / / / / / / / / / / / / /	10 2 ¬a, l ¬c, ✓	b, d, ¬e	

3. [3 points] Use the DPLL algorithm with the rules as described above to check whether the following formula in CNF is satisfiable

Clause 1: $\{a, b\}$

Clause 2: $\{\neg b, c\}$

Clause 3: $\{\neg a, \neg c\}$

Clause 4: $\{b,c\}$

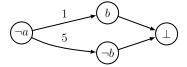
Clause 5: $\{a, \neg b\}$

Clause 6: $\{\neg b, \neg c\}$

Solution

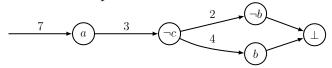
Step	1	2	3
Decision Level	0	1	1
Assignment	-	$\neg a$	$\neg a, \neg b$
Cl. 1: a, b	a, b	b	{} X
Cl. 2: $\neg b, c$	$\neg b, c$	$\neg b, c$	✓
Cl. 3: $\neg a, \neg c$	$\neg a, \neg c$	✓	✓
Cl. 4: b, c	b, c	b, c	c
Cl. 5: $a, \neg b$	$a, \neg b$	$\neg b$	✓
Cl. 6: $\neg b, \neg c$	$\neg b, \neg c$	$\neg b, \neg c$	✓
BCP	-	$\neg b$	-
PL	-	-	-
Decision	$\neg a$	-	-

Conflict in step 3



	,			
Step	4	5	6	7
Decision Level	0	0	0	0
Assignment	-	a	$a, \neg c$	$a, \neg c, \neg b$
Cl. 1: a, b	a, b	1	1	✓
Cl. 2: $\neg b, c$	$\neg b, c$	$\neg b, c$	$\neg b$	✓
Cl. 3: $\neg a, \neg c$	$\neg a, \neg c$	$\neg c$	1	✓
Cl. 4: b, c	b, c	b, c	b	{} X
Cl. 5: $a, \neg b$	$a, \neg b$	✓	1	✓
Cl. 6: $\neg b, \neg c$	$\neg b, \neg c$	$\neg b, \neg c$	1	✓
Cl. 7: a	a	✓	1	✓
BCP	a	$\neg c$	$\neg b$	-
PL	-	-	-	-
Decision	-	-	-	UNSAT
O 0: 1 : 1	_			

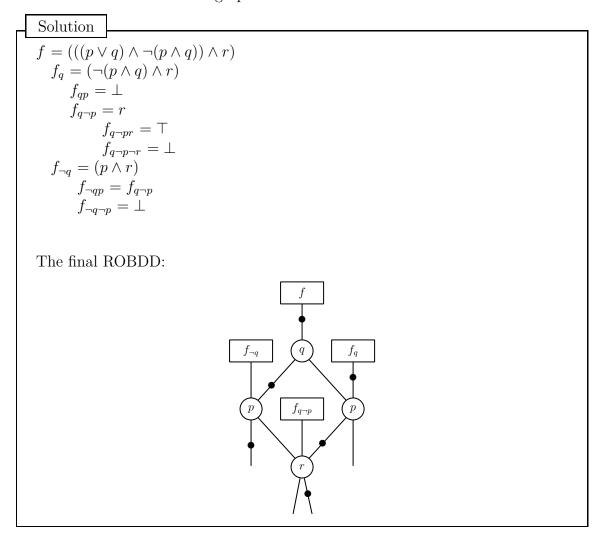
Conflict in step 7



4. [3 points] Construct a reduced ordered binary decision diagram (ROBDD) for the formula

$$f = (p \lor q) \land \neg (p \land q) \land r$$

using variable order q . Use complemented edges and a node for true as the only constant node. To simplify drawing, you may assume that dangling edges point to the constant node. Write down all cofactors that you compute to obtain the final result and mark them in the graph.



5. [3 points] Construct a reduced ordered binary decision diagram (ROBDD) for the formula

$$(a \vee \neg b) \wedge \neg (c \vee d) \vee (a \wedge b),$$

using variable order a < b < c < d. Use complemented edges and a node for true as the only constant node. To simplify drawing, you may assume that dangling edges point to the constant node. Write down all cofactors that you compute to obtain the final result and mark them in the graph.

