Lecture Notes for

Logic and Computability

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Natural Deduction for Predicate Logic

In this chapter, we will discuss the natural deduction calculus for predicate logic. The proofs are formed by using the rules that we already discussed for propositional logic and additionally we add new rules for the quantifiers and equality. As in the natural deduction calculus for propositional logic, we will discuss *introduction* and *elimination* rules for the *quantifiers* and the *equality* predicate.

6.1 Natural Deduction Rules

6.1.1 Proof Rules for Universal Quantification

The '∀-Elimination' Rule

First we discuss the rule for eliminating ∀:

$$\frac{\forall x\ \phi}{\phi\left[t/x\right]}\ \forall\mathrm{e}$$

The rule says that if $\forall x\ \phi$ is true, then we could replace the x in ϕ with any term t (with the side condition for substitution that t has to be free for x in ϕ) and conclude that $\phi\left[t/x\right]$ is also true. Recall that $\phi\left[t/x\right]$ is obtained by replacing all free occurrences of x in ϕ by t. Since ϕ is assumed to be true for all x, then ϕ should also be true for any term t.

The '∀-Introduction' Rule

Now, let us take a look at the rule for introducing a ∀:

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad\phi\left[x_0/x\right] \end{array}}\quad x_0 \text{ fresh}}{\forall x\ \phi}\ \forall\mathrm{i}$$

This rule is a bit more complicated. Just like we have seen in propositional logic, it is composed of a proof box. The box used in this rule defines the scope of the *freshly introduced variable* x_0; i.e., the box limits the usage of x_0 to this box only. *The proof-rule tells us, that if we are starting with a* ***fresh variable*** *x_0 and we are able to prove some formula $\phi\left[x_0/x\right]$, we are able to derive $\forall x\ \phi$.*

Note, that x_0 has to be a fresh variable, that has never been used outside the box. As it is a fresh variable, we can consider it as an arbitrary term, of which we did not assume anything before. Thus any other term would also work in its place (e.g., x_1, y_2, t, ...). The rule says that if you can prove ϕ about an x_0 that is free and therefore is not special in any way, then you could prove it for any x whatsoever, hence the conclusion $\forall x\ \phi$.

Example. Give the proof for the following sequent:

$$\forall x\ \big(P(x) \to Q(x)\big), \forall x\ P(x)\ \vdash\ \forall x\ Q(x).$$

Solution.

1.		$\forall x\ \big(P(x) \to Q(x)\big)$	prem.
2.		$\forall x\ P(x)$	prem.
3.	x_0	$P(x_0) \to Q(x_0)$	∀e 1
4.		$P(x_0)$	∀e 2
5.		$Q(x_0)$	→ e 3,4
6.		$\forall x\ Q(x)$	∀i 3-5

Example. Give the proof for the following sequent:

$$\forall x \forall y\ P(x, y)\ \vdash\ \forall a \forall b\ P(a, b)$$

1.		$\forall x \forall y\ P(x, y)$	prem.
2.	x_0	$\forall y\ P(x_0, y)$	∀e 1
3.	y_0	$P(x_0, y_0)$	∀e 2
4.		$\forall b\ P(x_0, b)$	∀i 3
5.		$\forall a \forall b\ P(a, b)$	∀i 4

The structure of this proof is guided by the fact that the conclusion is a $\forall$ formula, therefore the application of the $\forall i$ rule is needed. So we set up the box controlling the scope of x_0, and we need to prove $Q(x_0)$ inside the box in order to be able to conclude $\forall x\ Q(x)$ outside of the box. Using $\forall e$, we get the two instances of the premises $P(x_0)$ and $P(x_0) \to Q(x_0)$ used to prove $Q(x_0)$.

Example. Give the proof for the following sequent:

$$\forall x\ \big(\neg P(x) \to Q(x)\big), \neg Q(t) \ \vdash\ P(t)$$

Solution.

1.	$\forall x\ \big(\neg P(x) \to Q(x)\big)$	prem.
2.	$\neg Q(t)$	prem.
3.	$\neg P(t) \to Q(t)$	$\forall$e 1
4.	$\neg\neg P(t)$	MT 3,2
5.	$P(t)$	$\neg\neg$e 4

Note that if you apply the $\forall$e rule, you can use for the substitution any term t (free for x in ϕ) which is helpful in your current proof.

6.1.2 Proof Rules for Existential Quantification

The '∃-Introduction' Rule

The $\exists i$ rule is simply:

$$\frac{\phi\,[t/x]}{\exists x\ \phi}\ \exists\text{i}$$

The rule tells us, that if $\phi\,[t/x]$ is true, we can conclude $\exists x\ \phi$. As $\exists x$ only asks for ϕ to be true for some term t (naturally, we impose the side condition that t be free for x in ϕ).

Example. Give the proof for the following sequent:

$$\forall x\ \big(P(x) \to Q(x)\big) \ \vdash\ \exists y\ \big(P(y) \to Q(y)\big)$$

Solution.

1.	$\forall x\ \big(P(x) \to Q(x)\big)$	prem.
2.	$P(t) \to Q(t)$	$\forall$e 1
3.	$\exists y\ \big(P(y) \to Q(y)\big)$	$\exists$i 2

Example. Give the proof for the following sequent:

$$\forall x\ \big(P(x) \wedge Q(x)\big) \ \vdash\ \forall x\ \big(P(x) \vee Q(x)\big)$$

Solutions.

1.	$\forall x\ \big(P(x) \wedge Q(x)\big)$	prem.
2.	$P(x_0) \wedge Q(x_0)$	$\forall$e 1
3.	$P(x_0)$	$\wedge$e$_1$ 2
4.	$P(x_0) \vee Q(x_0)$	$\vee$i$_1$ 3
5.	$\exists x\ \big(P(x) \vee Q(x)\big)$	$\exists$i 4

The '∃-Elimination' Rule

The rule for eliminating an $\exists$ relates to the already known $\vee$e. The $\exists e$ rule is defined as follows:

$$\frac{\exists x\ \phi \qquad \boxed{\begin{array}{ll} x_0 & \\ & \phi\left[x_0/x\right] \text{ ass.} \\ & \vdots \\ & \chi \end{array}}\ x_0 \text{ fresh}}{\chi}\ \exists\text{e}$$

Just like when eliminating $\vee$, we need to make a case analysis. As $\exists x\ \phi$ is assumed to be true, we know that ϕ is true for at least one 'value' of x, but we do not know for which value. So we do a case analysis over all those possible values by using a **fresh variable** x_0 representing all of those values. So if $\phi\left[x_0/x\right]$ allows us to prove some χ (that does not contain x_0), χ can be deduced outside of the box. The box is controlling two things: the scope of x_0 and also the scope of the assumption $\phi[x_0/x]$.

Example. Give the proof for the following sequent:

$$\forall x\ \big(P(x) \rightarrow Q(x)\big), \exists x\ P(x)\ \vdash\ \exists x\ Q(x)$$

Solution.

1.		$\forall x\ \big(P(x) \rightarrow Q(x)\big)$	prem.
2.		$\exists x\ P(x)$	prem.
3.	x_0	$P(x_0)$	ass.
4.		$P(x_0) \rightarrow Q(x_0)$	$\forall$e 1
5.		$Q(x_0)$	$\rightarrow$ e 4,3
6.		$\exists x\ Q(x)$	$\exists$i 5
7.		$\exists x\ Q(x)$	$\exists$e 2,3-6

The motivation for introducing the box in line 3 of this proof is the existential quantifier in the premise $\exists x P(x)$ which has to be eliminated. In line 4 we eliminate the $\forall$ from line 1. Now, we can extract $Q(x_0)$ using line 4 and line 3. In line 6 we introduce an $\exists$ and substitute the x_0 again with an x.

As the formula in line 6 does not contain x_0 any more, we now may close the box in accordance to our ∃e rule. To conclude our ∃e, which we started with the box at line 3, in line 7 we need to rewrite the same formula as in line 6.

Example. Consider the following proof and analyse the error made in this proof:

1.		$\forall x \left(P(x) \to Q(x)\right)$	prem.
2.		$\exists x\ P(x)$	prem.
3.	x_0	$P(x_0)$	ass.
4.		$P(x_0) \to Q(x_0)$	∀e 1
5.		$Q(x_0)$	→ e 4,3
6.		$Q(x_0)$	∃e 2,3-5
7.		$\exists x\ Q(x)$	∃i 6

Solution. Line 6 allows the fresh variable x_0 to escape the scope of the box which declares it. This is illegal. Therefore, the $\exists i$ rule has to be applied already inside of the box like in the proof above.

Boxes may also be nested within each other. But we need to be careful, on where our scopes begin and where they end. To understand the concept of multiple boxes, we take a look at another interesting example.

Example. Give the proof for the following sequent:

$$\exists x\ P(x), \forall x\ \forall y\ \left(P(x) \to Q(y)\right)\ \vdash\ \forall y\ Q(y)$$

Solution.

1.			$\exists x\ P(x)$	prem.
2.			$\forall x\ \forall y\ \left(P(x) \to Q(y)\right)$	prem.
3.	y_0			
4.		x_0	$P(x_0)$	ass.
5.			$\forall y\ \left(P(x_0) \to Q(y)\right)$	∀e 2
6.			$P(x_0) \to Q(y_0)$	∀e 5
7.			$Q(y_0)$	→ e 6,4
8.			$Q(y_0)$	∃e 1,4-7
9.			$\forall y\ Q(y)$	∀i 3-8

In this example , the first premise id an ∃ formula, which requires an $\exists e$ to be of any use. The conclusion is an ∀ formula, which requires the application of the $\forall i$ rule. Therefore, this proof has to boxes. The outer box from 3-8 is for introducing ∀, whereas the inner box from 4-7 is for eliminating the ∃ from line 1. We need to declare for both boxes fresh variables. To keep it simple, we will substitute y_0 for y for the outer box and x_0 for x for the inner box. Note again, that it is important to not use x_0 and y_0 outside of there boxes.

Example. Give the proof for the following sequent:

$$\forall x\ \big(P(x) \land Q(x)\big) \ \vdash\ \forall x\ P(x) \land \forall x\ Q(x)\big)$$

Solution.

1.		$\forall x\ \big(P(x) \land Q(x)\big)$	prem.
2.	x_0	$P(x_0) \land Q(x_0)$	$\forall$e 1
3.		$P(x_0)$	$\land$e$_1$ 2
4.		$\forall x\ P(x)$	$\forall$i 2-3
5.	y_0	$P(y_0) \land Q(y_0)$	$\forall$e 1
6.		$Q(y_0)$	$\land$e$_2$ 5
7.		$\forall x\ Q(x)$	$\forall$i 5-6
8.		$\forall x\ P(x) \land \forall x\ Q(x)$	$\land$i 4,7

Example. Give the proof for the following sequent:

$$\exists x\ P(x) \ \vdash\ \neg\forall x\ \neg P(x)$$

Solution.

1.		$\exists x\ P(x)$	prem.
2.		$\forall x\ \neg P(x)$	ass.
3.	x_0	$P(x_0)$	ass.
4.		$\neg P(x_0)$	$\forall$ 2
5.		$\bot$	$\neg$e 3,4
6.		$\bot$	$\exists$e 1,4-5
7.		$\neg\forall x\ \neg P(x)$	$\neg$i 2-6

Example. Give the proof for the following sequent:

$$\forall x\ P(a,x,x), \forall x \forall y \forall z\ \big(P(x,y,z) \to P(f(x),y,f(z))\big) \ \vdash\ P\big((f(a),a,f(a)\big)$$

Solution.

1.	$\forall x\ P(a,x,x)$	prem.
2.	$\forall x \forall y \forall z\ \big(P(x,y,z) \to P(f(x),y,f(z))\big)$	prem.
3.	$P(a,a,a)$	$\forall$e 1
4.	$\forall y \forall z\ \big(P(a,y,z) \to P(f(a),y,f(z))\big)$	$\forall$e 2
5.	$\forall z\ \big(P(a,a,z) \to P(f(a),a,f(z))\big)$	$\forall$e 4
6.	$P(a,a,a) \to P(f(a),a,f(a))$	$\forall$e 5
7.	$P\big((f(a),a,f(a)\big)$	$\to$ e 6,3

Example. Give the proof for the following sequent:

$$\exists y \forall x \big(s(x) = t(y)\big) \;\vdash\; \forall x \exists y \, \big(s(x) = t(y)\big)$$

Solution.

1.		$\exists y \forall x \big(s(x) = t(y)\big)$	prem.
2.	y_0	$\forall x \big(s(x) = t(y_0)\big)$	ass.
3.	x_0	$s(x_0) = t(y_0)$	$\forall$e 2
4.		$\exists y \, \big(s(x_0) = t(y)\big)$	$\exists$i 3
5.		$\forall x \exists y \, \big(s(x) = t(y)\big)$	$\forall$i 3-4
6.		$\forall x \exists y \, \big(s(x) = t(y)\big)$	$\exists$e 1,2-5

Example. Give the proof for the following sequent:

$$\neg \forall x \, \big(P(x) \wedge Q(x) \wedge R(y)\big) \;\vdash\; \exists x \, \neg \big(P(x) \wedge Q(x) \wedge R(y)\big)$$

Solution.

1.	$\neg \forall x \, \big(P(x) \wedge Q(x) \wedge R(y)\big)$	prem.
2.	$P(t) \wedge Q(t) \wedge R(y)$	ass.
3.	$\forall x \, \big(P(x) \wedge Q(x) \wedge R(y)\big)$	$\forall$i 2
4.	$\bot$	$\neg$e 1,3
5.	$\neg \big(P(t) \wedge Q(t) \wedge R(y)\big)$	$\neg$i 2-4
6.	$\exists x \, \neg \big(P(x) \wedge Q(x) \wedge R(y)\big)$	$\exists$i 5

Example. Give the proof for the following sequent:

$$\exists x \, \neg \big(P(x) \wedge Q(x) \wedge R(y)\big) \;\vdash\; \neg \forall x \, \big(P(x) \wedge Q(x) \wedge R(y)\big)$$

Solution.

1.		$\exists x \, \neg \big(P(x) \wedge Q(x) \wedge R(y)\big)$	prem.
2.		$\forall x \, \big(P(x) \wedge Q(x) \wedge R(y)\big)$	ass.
3.	t	$\neg \big(P(t) \wedge Q(t) \wedge R(y)\big)$	ass.
4.		$P(t) \wedge Q(t) \wedge R(y)$	$\forall$e 2
5.		$\bot$	$\neg$e 3,4
6.		$\bot$	$\exists$e 1,3-5
7.		$\neg \forall x \, \big(P(x) \wedge Q(x) \wedge R(y)\big)$	$\neg$i 2-6

6.1.3 Quantifier Equivalences

To train to perform your own natural deduction proofs, you can consider creating proofs for the most commonly used quantifier equivalences. The proofs are interesting, because most of them involve several quantifications over more than just one variable and your proofs will have nested boxes.

For example, you can proof the following equivalences by creating a proof for each direction:

$$\neg\forall x\ \phi \equiv \exists x\ \neg\phi$$
$$\neg\exists x\ \phi \equiv \forall x\ \neg\phi$$
$$\neg\forall x\ \neg\phi \equiv \exists x\ \phi$$
$$\neg\exists x\ \neg\phi \equiv \forall x\ \phi$$

Example. Proof the following quantifier equivalence:

$$\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$$

Solution. For both directions, we create a proof:

$\forall x\ \neg P(x)\ \vdash\ \neg\exists x\ P(x)$

1.		$\forall x\ \neg P(x)$	prem.
2.		$\exists x\ P(x)$	ass.
3.	t	$P(t)$	ass.
4.		$\neg P(t)$	$\forall$e 1
5.		$\bot$	$\neg$e 3,4
6.		$\bot$	$\exists$e 2,3-5
7.		$\neg\exists x\ P(x)$	$\neg$i 2-6

$\neg\exists x\ P(x)\ \vdash\ \forall x\ \neg P(x)$

1.		$\neg\exists x\ P(x)$	prem.
2.	t		
3.		$P(t)$	ass.
4.		$\exists x\ P(x)$	$\exists$i 3
5.		$\bot$	$\neg$e 1,4
6.		$\neg P(t)$	$\neg$i 3-5
7.		$\forall x\ \neg P(x)$	$\forall$i 2-6

6.1.4 Counterexamples

If a sequent is not valid, there is no natural deduction proof for such a sequent. In such cases, we construct a counter-example that proofs the sequent to be invalid. As discussed in the chapters addressing propositional logic, *a counter-example is a model, that satisfies all the premises but does not satisfy the conclusion.*

Example. Show that the following sequent is invalid by constructing a counter-example for it:

$$\exists x\ \big(P(x) \to S\big)\ \vdash\ \exists x\ P(x) \to S$$

Solution. We define the following model $\mathcal{M}$ that serves as a counterexample:

- $\mathcal{A} = \{a, b\}$
- $P^{\mathcal{M}} = \{a\}$
- $S^{\mathcal{M}} = \bot$

Note, that the $\exists$ operator binds strongest. Therefore the scope of $\exists x$ is only $P(x)$ in the conclusion.

To show, that $\mathcal{M}$, we first show that $\mathcal{M}$ violates the conclusion, and second that it satisfies the premise.

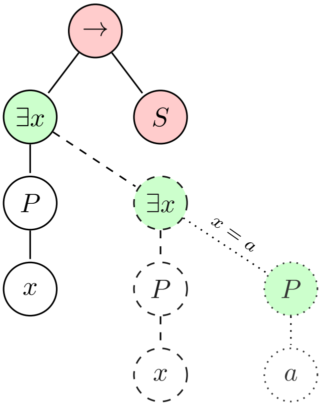

Figure 6.1: Syntax tree of the conclusion. Red nodes are *false*, green nodes are *true*. Dotted and dashed nodes are subtrees with substituted variables.

Showing that $\mathcal{M} \not\vDash \exists x\ P(x) \to S$*:* First we draw the syntax tree as shown in Figure 6.1. Then we evaluate the nodes. To satisfy the $\exists x$ node in our formula, we need to find at least one value for x that $P(x)$ true. When substituting $\left[a/x\right]$, we get that $P^{\mathcal{M}}(a) = \mathit{true}$, which also makes the $\exists x$ node true. The predicate S always evaluates to false. Therefore, the implication results in a $\perp$, thus making the conclusion false.

Showing that $\mathcal{M} \vDash \exists x\ \big(P(x) \to S\big)$*:* Again, we draw the syntax tree as shown in Figure 6.2 and evaluate the nodes. In order for the $\exists x$ node to become true, we need to fine a value for x that makes the implication node true. Again we first try to substitute $\left[a/x\right]$, which results in a true $P(x)$, but in a false implication. If we, however, substitute $\left[b/x\right]$, $P(x)$ evaluates to false and thus making the implication true. Thus also our $\exists x$ is true and therefore also the whole premise.

We now have shown, that our given model $\mathcal{M}$ satisfies the premise, but not the conclusion. Therefore $\mathcal{M}$ is a counterexample. If we would not have been able to find a counterexample, the sequence must have been true.

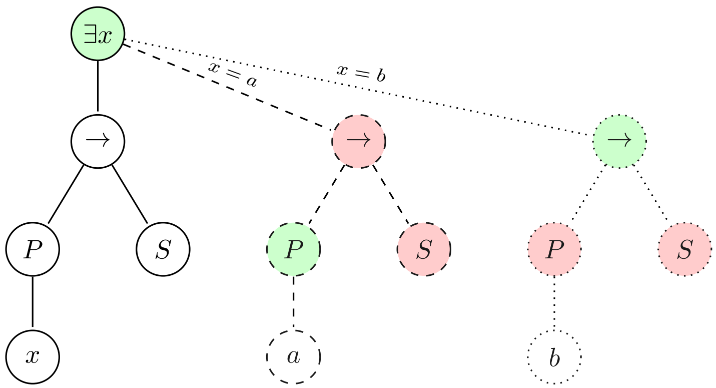

Figure 6.2: Syntax tree of the premise. Red nodes are *false*, green nodes are *true*. Dotted and dashed nodes are subtrees with substituted variables.

Declaration of Sources

Chapter 7 was based on the following book. Michael Huth, Mark Dermot Ryan: *Logic in Computer Science: Modelling and Reasoning about Systems.* June 2004. Cambridge University Press. ISBN:978-0-521-54310-1