Lecture Notes for

Logic and Computability

Course Number: IND04033UF

Contact

Bettina Könighofer
Institute for Applied Information Processing and Communications (IAIK)
Graz University of Technology, Austria
bettina.koenighofer@iaik.tugraz.at

Table of Contents

9 Combinational Equivalence Checking **3**
9.1 Translation of a Circuit into a Formula 4
9.2 Relations between Satisfiability, Validity, Equivalence and Entailment 5
9.3 Normal Forms 7
9.4 Tseitin Encoding 9
9.5 CEC Example 11

over

9

Combinational Equivalence Checking

In this chapter we discuss combinational equivalence checking (CEC) based on Boolean satisfiability (SAT). CEC plays an important role in the design of electronic systems such as integrated circuits and printed circuit boards. Its immediate application is verifying functional equivalence of combinational circuits after the application of synthesis and optimization tools. In a typical scenario, there are two structurally different implementations of the same design, and the problem is to prove their functional equivalence.

The standard approach for checking the equivalence of two circuits is to reduce the equivalence problem to SAT and using a SAT solver to solve the problem instance. Though the problem is NP-complete, most verification instances can be solved in reasonable space and time resources due to the high efficiency of modern SAT solvers.

Overview of CEC based on Satisfiablilty

In the following, we give the overview of the algorithm to check for the equivalence of two circuits. In the reminder of this chapter, we will discuss the individual steps in detail.

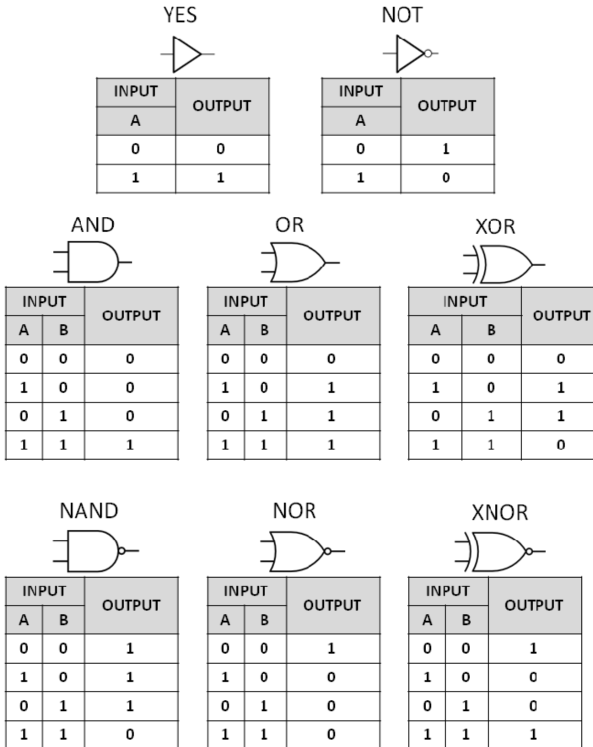
Algorithm - Decide equivalence of combinational circuits: Let C_1 and C_2 denote the two combinational circuits. In order to check whether C_1 and C_2 are equivalent, one has to perform the following steps:

1. Encode C_1 and C_2 into two propositional formulas φ_1 and φ_2.
2. Compute the Conjunctive Normal Form (CNF) of $\varphi_1 \oplus \varphi_2$, using Tseitin encoding; i.e., $CNF(\varphi_1 \oplus \varphi_2)$.

3. Give the formula $CNF(\varphi_1 \oplus \varphi_2)$ to a SAT solver and check for satisfiability.
4. C_1 and C_2 are equivalent if and only if $CNF(\varphi_1 \oplus \varphi_2)$ is UNSAT.

9.1 Translation of a Circuit into a Formula

Each gate of the circuit can be represented via a propositional formula over its inputs and outputs. The following figure lists a summary of the common Boolean logic gates with their symbols and truth tables.



YES

INPUT A	OUTPUT
0	0
1	1

NOT

INPUT A	OUTPUT
0	1
1	0

AND

INPUT A	INPUT B	OUTPUT
0	0	0
1	0	0
0	1	0
1	1	1

OR

INPUT A	INPUT B	OUTPUT
0	0	0
1	0	1
0	1	1
1	1	1

XOR

INPUT A	INPUT B	OUTPUT
0	0	0
1	0	1
0	1	1
1	1	0

NAND

INPUT A	INPUT B	OUTPUT
0	0	1
1	0	1
0	1	1
1	1	0

NOR

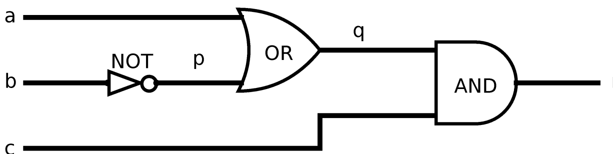
INPUT A	INPUT B	OUTPUT
0	0	1
1	0	0
0	1	0
1	1	0

XNOR

INPUT A	INPUT B	OUTPUT
0	0	1
1	0	0
0	1	0
1	1	1

Figure 9.1: Boolean logic gates.

Example. Translate the following circuit C into a propositional formula.



The inputs of C are denoted by a, b, and c and the output is denoted by r. We assign temporary variable names to the inner wires; in this case we use p and q. Using these variables, we can create the propositional formula over the inputs and the output of C.

$$\begin{aligned} r &= q \wedge c \\ &= (a \vee p) \wedge c \\ &= (a \vee \neg b) \wedge c \end{aligned} \tag{9.1}$$

9.2 Relations between Satisfiability, Validity, Equivalence and Semantic Entailment

In the first chapter, we discussed the notions of satisfiability, validity, equivalence, and semantic entailment. *All of this notions can be reduced to each other. Therefore, only one decision procedure is needed to decide all notions.* This is in particularly important, since for the question of satisfiability there exists tools – SAT solvers – that are able to decide most practical instances highly efficient, even though the problem of deciding satisfiability is NP-complete. Therefore, in order to answer the questions of validity, semantic entailment, or equivalence, these questions will be reduced to the question of satisfiability most of the time.

In this section, we discuss the relations between the notions.

Duality of Satisfiability and Validity

A formula φ is valid, if and only if, $\neg\varphi$ is not satisfiable.

Consider the formula $\varphi = (x \vee \neg x)$. This formula is valid, i.e., all rows in the truth table would evaluate to *true*. The negation of φ is the following: $\neg\varphi = \neg(x \vee \neg x) = \neg x \wedge x$, which is not satisfiable, i.e., all rows the truth table would evaluate to *false*.

A formula φ is satisfiable, if and only if, $\neg\varphi$ is not valid.

If φ is satisfiable, there is at least one model that makes the formula true. If we negate the formula, these models make the negated formula false, and therefore, the negated formula cannot be valid.

using \ solving	φ satisfiable?	φ valid?
Satisfiability	✓	$\neg\varphi$ not satisfiable?
Validity	$\neg\varphi$ not valid?	✓

Overview over all Relations

using \ solving	φ satisfiable?	φ valid?	$\varphi \models \psi$?	$\varphi \equiv \psi$?
Satisfiability	✓	$\neg\varphi$ not satisfiable?	$\varphi \land \neg\psi$ not satisfiable?	$\varphi \oplus \psi$ not satisfiable?
Validity	$\neg\varphi$ not valid?	✓	$\varphi \to \psi$ valid?	$\varphi \leftrightarrow \psi$ valid?
Entailment	$\top \not\models \neg\varphi$?	$\top \models \varphi$?	✓	$\varphi \models \psi$ and $\psi \models \varphi$?
Equivalence	$\varphi \not\equiv \bot$?	$\varphi \equiv \top$?	$\varphi \to \psi \equiv \top$?	✓

Row 1: Decide semantic entailment using satisfiability. The question whether $\varphi \models \psi$ can be decided by checking $\varphi \land \neg\psi$ not satisfiable. If there does not exist a counterexample, i.e., a model that makes φ true and makes ψ false, then we have semantic entailment.

Row 1: Decide equivalence using satisfiability. The question whether $\varphi \equiv \psi$ can be decided by checking $\varphi \oplus \psi$ not satisfiable. If there does not exists a model, that makes one sub-formula true, and the other sub-formula false, then the two formulas are semantically equivalent.

Row 2: Decide semantic entailment using validity. The question whether $\varphi \models \psi$ can be decided by checking $\varphi \to \psi$ is valid. For an implication to hold, all models that satisfy φ also have to satisfy ψ. This is also the requirement for $\varphi \models \psi$ to be true.

Row 2: Decide equivalence using validity. The question whether $\varphi \equiv \psi$ can be decided by checking $\varphi \leftrightarrow \psi$ is valid. This holds, since the formula $\varphi \leftrightarrow \psi$ only evaluates to true under models that either make both sub-formulas true, or both false.

Row 3: Decide satisfiability using semantic entailment. The question whether φ is satisfiable can be decided by checking $\top \not\models \neg\varphi$. If not all models satisfy $\neg\varphi$, then there exists a model that satisfies φ, and therefore φ must be satisfiable.

Row 3: Decide validity using semantic entailment. The question whether φ is valid can be decided by checking $\top \models \varphi$. If all models satisfy φ, then φ must be valid.

Row 3: Decide equivalence using entailment. The question whether $\varphi \equiv \psi$ is valid can be decided by checking $\varphi \models \psi$ and $\psi \models \varphi$. If all models that satisfy φ also satisfy ψ and all models that satisfy ψ also satisfy φ, then φ and ψ are satisfied by exactly the same models and are equivalent.

Row 4: Decide satisfiability using equivalence. The question φ is satisfiable can be decided by checking $\varphi \not\equiv \bot$. If it is not true, that φ evaluates to false under all models, then φ is satisfiable.

Row 4: Decide validity using equivalence. The question whether φ is

valid can be decided by checking $\varphi \equiv \top$. If it is true, that φ evaluates to true under all models and is valid.

Row 4: Decide entailment with equivalence. The question whether $\varphi \models \psi$ can be decided by checking $\varphi \rightarrow \psi \equiv \top$. If all models that satisfy φ also satisfy ψ, then we have semantic entailment.

9.3 Normal Forms

For ease of implementation, most SAT solvers operate on formulas given in conjunctive normal form (CNF). In this section we discuss the two standard normal forms: the disjunctive normal form (DNF) and the conjunctive normal form (CNF).

Terminology

Let φ be a propositional formula defined over Boolean variables $x_1, ..., x_n$. We introduce the following symbols and terminology to define CNF and DNF.

- A *literal* is a variable x_i or its negation.
- A literal is called *positive* if it is just a variable. A literal is called *negative* if it is the negation of a variable.
- A *clause* is a disjunction of literals, e.g., $x_1 \vee x_2$.
- A *cube* is a conjunction of literals, e.g., $x_1 \wedge x_2$.

Disjunctive Normal Form (DNF)

A formula in DNF is a *disjunction of cubes*, e.g., $(p \wedge q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (\neg q)$.

Any formula in propositional logic can be converted to the disjunctive normal form. One simple way to do so is to use the truth table of the formula. For every model/line that delivers a 1, a *conjunction* of all literals of that lines is formed, i.e., variables which are assigned 1 in the line are not negated and variables which are assigned 0 are negated. These conjunctions per line form the cubes of the DNF formula. By connecting all cubes with disjunctions, one obtains the final DNF formula.

Example. Given the formula $\varphi := \neg a \vee \neg(b \rightarrow c)$. Compute its representation in DNF using its truth table.

a	b	c	$\neg a \vee \neg(b \to c)$
F	F	F	T
F	F	T	T
F	T	F	T
F	T	T	T
T	F	F	F
T	F	T	F
T	T	F	T
T	T	T	F

The resulting formula in DNF is

$$(\neg a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c) \vee (\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge b \wedge c) \vee (a \wedge b \wedge \neg c).$$

Conjunctive Normal Form (CNF)

A formula in CNF is a *conjunction of clauses*, e.g., $(p \vee q \vee r) \wedge (\neg p \vee q \vee r)$.

To convert the formula to CNF, we consider the models/lines that make the formula *false*. For every line that delivers a 0, a *disjunction* of the *negated literals* of that lines is formed, i.e., variables which are assigned 1 in the line are negated and variables which are assigned 0 are not negated. These disjunctions per line form the clauses of the CNF formula. By connecting all clauses with conjunctions, one obtains the final CNF formula.

Example. Given the formula $\varphi := \neg a \vee \neg(b \to c)$. Compute its representation in CNF using its truth table.

a	b	c	$\neg a \vee \neg(b \to c)$
F	F	F	T
F	F	T	T
F	T	F	T
F	T	T	T
T	F	F	F
T	F	T	F
T	T	F	T
T	T	T	F

The resulting formula in DNF is

$$(\neg a \vee b \vee c) \wedge (\neg a \vee b \vee \neg c) \wedge (\neg a \vee \neg b \vee \neg c).$$

Example. Provide the formula in CNF and in DNF that is represented by the following truth table.

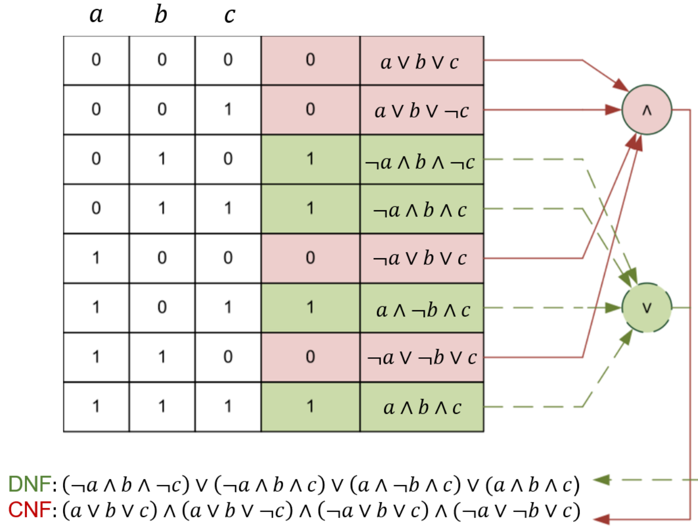


Figure 9.2: DNF/CNF example.[1]

9.4 Tseitin Encoding

Propositional formulas need to be converted into CNF before they can be given to a SAT solver. The disadvantage for using truth tables for the conversion is that for some formulas, the resulting representation in CNF might be *exponentially* larger than the original formula. Therefore, we do not generate a CNF that is equivalent, but instead perform a transformation that *only preserves satisfiability*. We call such a formula that is not equivalent to the original formula but preserves satisfiability *equisatisfiable*.

Definition - Equisatisfiability. Two propositional formulas φ and ψ are *equisatisfiable* if and only if either *both are satisfiable* or *both are unsatisfiable*.

One way to perform the conversion of a propositional formula into an equisatisfiable formula in CNF is the *Tseitin's method*, which only results in a *linear* blow-up.

The Tseitin Algorithm

Tseitin's method takes the syntax tree for a propositional formula φ as input. An internal node in this tree is a Boolean connective while a leaf is a Boolean variable. The algorithm traverses the tree, beginning with the leaves, and associates a *fresh variable* – called tseitin variable – to each node, i.e., to each subformula. By traversing through the tree, the algorithm collects a set of clauses.

[1] Taken from `https://de.wikipedia.org/wiki/Konjunktive_Normalform#/media/Datei:Knf+dnf.svg` by JensKhol, licensed under CC-BY-SA-2.0

We will explain the Tseitin's method for formulas that are restricted to the Boolean connectives $\wedge$, $\vee$, and $\neg$.

Step 1: Assigning tseitin variables to all nodes in the parse tree / to each subformula.

Example. Consider the formula $\varphi := ((p \vee q) \wedge r) \vee \neg p$. For each sub-formula, we introduce a tseitin variable.

$$\underbrace{(\underbrace{(\underbrace{p \vee q}_{x_1}) \wedge r}_{x_2}) \vee \underbrace{\neg p}_{x_3}}_{x_\varphi}$$

Step 2: Add new clauses for each tseitin variable.

Each new tseitin variable represents either a $\neg$, $\wedge$, or $\vee$ node in the parse tree, i.e., sub-formula of the form $\neg\varphi$, $\varphi \wedge \psi$, or $\varphi \vee \psi$. The *Tseitin-Rewriting Rules* define, which clauses are associated with which type of node / subformula.

Tseitin-Rewrite rules:

For each node, let x be the tseitin variable, and let p and q be the new variables for the two subformulas of the node.

- $\wedge$ node: for $x \leftrightarrow (p \wedge q)$ introduce the clauses $(\neg x \vee p) \wedge (\neg x \wedge q) \wedge (\neg p \vee \neg q \vee x)$.
- $\vee$ node: for $x \leftrightarrow (p \vee q)$ introduce the clauses $(\neg p \vee x) \wedge (\neg q \vee x) \wedge (\neg x \vee p \vee q)$.
- $\neg$ node: for $x \leftrightarrow \neg p$ introduce the clauses $(\neg x \vee \neg p) \wedge (p \vee x)$.

Finally, to obtain the final equisatisfiable formula in CNF, we connect the variable for the top level node of the formula and all generated clauses with conjunctions.

Example. Using the Tseitin algorithm and its rewriting rules, we would obtain the following clauses for the example above with $\varphi := ((p \vee q) \wedge r) \vee \neg p$:

$$\begin{aligned} CNF(\varphi) = \quad & (\neg p \vee x_1) \wedge (\neg q \vee x_1) \wedge (\neg x_1 \vee p \vee q) \\ & \wedge (\neg x_2 \vee x_1) \wedge (\neg x_2 \vee r) \wedge (\neg x_1 \vee \neg r \vee x_2) \\ & \wedge (\neg x_3 \vee \neg p) \wedge (p \vee x_3) \\ & \wedge (\neg x_2 \vee x_\varphi) \wedge (\neg x_3 \vee x_\varphi) \wedge (\neg x_\varphi \vee x_2 \vee x_3) \\ & \wedge x_\varphi \end{aligned} \tag{9.2}$$

Derivation of the Tseitin Rewriting Rules

Following we briefly justify the clauses that are generated for $\wedge$, $\vee$, and $\neg$ nodes.

- $x \leftrightarrow (p \wedge q)$ generates the clauses $(\neg x \vee p) \wedge (\neg x \vee q) \wedge (\neg p \vee \neg q \vee x)$ because:

$$\begin{array}{ll}
x \leftrightarrow (p \wedge q) & \\
(x \rightarrow (p \wedge q)) \wedge ((p \wedge q) \rightarrow x) & \mid \phi \leftrightarrow \psi \equiv (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi) \\
(x \rightarrow p) \wedge (x \rightarrow q) \wedge ((p \wedge q) \rightarrow x) & \mid \phi \rightarrow (\psi \wedge \chi) \equiv (\phi \rightarrow \psi) \wedge (\phi \rightarrow \chi) \\
(\neg x \vee p) \wedge (\neg x \wedge q) \wedge (\neg(p \wedge q) \vee x) & \mid \phi \rightarrow \psi \equiv \neg\phi \vee \psi \\
(\neg x \vee p) \wedge (\neg x \wedge q) \wedge (\neg p \vee \neg q \vee x) & \mid \neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi
\end{array}$$

- $x \leftrightarrow (p \vee q)$ generates the clauses $(\neg p \vee x) \wedge (\neg q \vee x) \wedge (\neg x \vee p \vee q)$ because:

$$\begin{array}{ll}
x \leftrightarrow (p \vee q) & \\
(x \rightarrow (p \vee q)) \wedge ((p \vee q) \rightarrow x) & \mid \phi \leftrightarrow \psi \equiv (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi) \\
(x \rightarrow (p \vee q)) \wedge (p \rightarrow x) \wedge (q \rightarrow x) & \mid \psi \vee \phi \rightarrow \chi \equiv (\phi \rightarrow \chi) \wedge (\psi \rightarrow \chi) \\
(\neg x \vee p \vee q) \wedge (\neg p \vee x) \wedge (\neg q \vee x) & \mid \phi \rightarrow \psi \equiv \neg\phi \vee \psi \\
(\neg p \vee x) \wedge (\neg q \vee x) \wedge (\neg x \vee p \vee q) & \mid \text{rearranging}
\end{array}$$

- $x \leftrightarrow \neg p$ generates the clauses $(\neg x \vee \neg p) \wedge (p \vee x)$ because:

$$\begin{array}{ll}
x \leftrightarrow \neg p & \\
(x \rightarrow \neg p) \wedge (\neg p \rightarrow x) & \mid \phi \leftrightarrow \psi \equiv (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi) \\
(\neg x \vee \neg p) \wedge (\neg\neg p \vee x) & \mid \phi \rightarrow \psi \equiv \neg\phi \vee \psi \\
(\neg x \vee \neg p) \wedge (p \vee x) & \mid \neg\neg\phi \equiv \phi
\end{array}$$

Summary: Tseitin Rewriting Rules

$$\begin{array}{rcl}
\chi \leftrightarrow (\varphi \vee \psi) & \Leftrightarrow & (\neg\varphi \vee \chi) \wedge (\neg\psi \vee \chi) \wedge (\neg\chi \vee \varphi \vee \psi) \\
\chi \leftrightarrow (\varphi \wedge \psi) & \Leftrightarrow & (\neg\chi \vee \varphi) \wedge (\neg\chi \vee \psi) \wedge (\neg\varphi \vee \neg\psi \vee \chi) \\
\chi \leftrightarrow \neg\varphi & \Leftrightarrow & (\neg\chi \vee \neg\varphi) \wedge (\varphi \vee \chi)
\end{array}$$

Properties of Tseitin Encoding

We make two observations about the set of clauses that is generated by the Tseitin procedure for a formula φ.

1. The number of variables and clauses is *linear* in the size of φ. We have thus avoided the exponential blowup we would have observed when constructing an equisatisfiable CNF formula $CNF(\varphi)$.
2. The constructed formula $CNF(\varphi)$ is equisatisfiable to φ: $CNF(\varphi)$ has a satisfying assignment if and only if there is a satisfying assignment for φ. We can obtain a satisfying assignment for φ from the satisfying assignment for $CNF(\varphi)$ by simply dropping the additional variables that the algorithm has introduced.

9.5 CEC Example

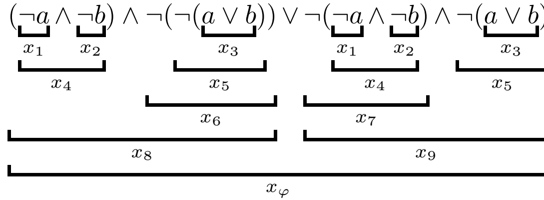
Example. Given are two formulas $\varphi_1 = \neg a \wedge \neg b$ and $\varphi_2 = \neg(a \vee b)$. Check whether φ_1 and φ_2 are semantically equivalent using the reduction to satisfiability.

We need to check, whether $\varphi_1 \oplus \varphi_2$ is not satisfiable, i.e, $\varphi_1 \equiv \varphi_2$ *if and only if $\varphi_1 \oplus \varphi_2$ is UNSAT*. Therefore, we perform the following steps.

- We construct the formula φ:

$$\begin{aligned}\varphi = \varphi_1 \oplus \varphi_2 &= (\neg a \wedge \neg b) \oplus \neg(a \vee b) \\ &= (\neg a \wedge \neg b) \wedge \neg(\neg(a \vee b)) \ \vee \ \neg(\neg a \wedge \neg b) \wedge \neg(a \vee b)\end{aligned} \tag{9.3}$$

- Next, the formula φ has to be transformed into a CNF formula by using Tseitin encoding.
  Tseitin-Step 1: Assigning new varibles to subformulas.



  Tseitin-Step 2: Apply rewriting rules.

$$\begin{aligned}
CNF(\varphi) = x_\varphi \wedge & \\
& (\neg x_8 \vee x_\varphi) \wedge (\neg x_9 \vee x_\varphi) \wedge (\neg x_\varphi \vee x_9 \vee x_8) \wedge && \mid \text{rewriting } x_\varphi \\
& (\neg x_9 \vee x_7) \wedge (\neg x_9 \vee x_5) \wedge (\neg x_7 \vee \neg x_5 \vee x_9) \wedge && \mid \text{rewriting } x_9 \\
& (\neg x_8 \vee x_4) \wedge (\neg x_8 \vee x_6) \wedge (\neg x_4 \vee \neg x_6 \vee x_8) \wedge && \mid \text{rewriting } x_8 \\
& (\neg x_7 \vee \neg x_4) \wedge (x_7 \vee x_4) \wedge && \mid \text{rewriting } x_7 \\
& (\neg x_6 \vee \neg x_5) \wedge (x_6 \vee x_5) \wedge && \mid \text{rewriting } x_6 \\
& (\neg x_5 \vee \neg x_3) \wedge (x_5 \vee x_3) \wedge && \mid \text{rewriting } x_5 \\
& (\neg x_4 \vee x_1) \wedge (\neg x_4 \vee x_3) \wedge (\neg x_1 \vee \neg x_3 \vee x_4) \wedge && \mid \text{rewriting } x_4 \\
& (\neg a \vee x_3) \wedge (\neg b \vee x_3) \wedge (\neg x_3 \vee a \vee b) \wedge && \mid \text{rewriting } x_3 \\
& (\neg x_2 \vee \neg b) \wedge (x_2 \vee b) \wedge && \mid \text{rewriting } x_2 \\
& (\neg x_1 \vee \neg a) \wedge (x_1 \vee a) && \mid \text{rewriting } x_1
\end{aligned}$$

- Finally, the formula $CNF(\varphi)$ is given to a SAT solver. If the SAT solver determines that $CNF(\varphi)$ is unsatisfiable, then $\varphi_1 \equiv \varphi_2$. If the SAT solver determines that $CNF(\varphi)$ is satisfiable, then $\varphi_1 \not\equiv \varphi_2$.