Logic and Computability

Lecture 4

Natural Deduction

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https://xkcd.com/1724/

Motivation – Natural Deduction

- Formalize Structure of Reasoning
  - Reasoning rules
    - Purely typographical / syntactic Rules
  - Deduce new knowledge
    - From given premises we deduce the conclusion

- Advantages
  - Watertight" Proofs
    - Automatically checkable
  - Automation for proof generation

- Basis for "Real Proofs"

Motivation

p
- The train is late.

q
- There are taxis at the station.

r
- John is late for the meeting.

Motivation

p • The train is late.

q • There are taxis at the station.

r • John is late for the meeting.

Knowledge:

- $p \land \neg q \rightarrow r$
- $\neg r$
- p

therefore →

q

Motivation

	Example 1	Example 2
p	The train is late.	Max is registered for LuB.
q	There are taxis at the station.	Max passes the exam.
r	John is late for the meeting.	Max gets a negative grade.

Knowledge:

- $p \wedge \neg q \rightarrow r$
- $\neg r$
- p

therefore →

q

Outline

- Proof Rules
  - Introduction Rules
  - Elimination Rules
- Soundness and Completeness
  - Proof the invalidity of sequences via counter examples

Sequents

$$\underbrace{\phi_1, \ \phi_2, \ \dots, \ \phi_n}_{\textbf{Premises}} \vdash \underbrace{\Psi}_{\textbf{Conclusion}}$$

$\vdash$ … single turnstile

read: „entails“

„proofs“

(Latex: \vdash)

Example 1

[Lecture] Give the definition of a sequent. Give an example of a sequent and name the parts the sequent consists of.

A sequent is an expression of the form

$$\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi.$$

$\varphi_1, \varphi_2, \ldots, \varphi_n$ are called premises. ψ is called the conclusion.
The premises entail the conclusion. This means that for any valid sequence, we can proof that the conclusion follows from the premises.

Example 2

[Lecture] Look at the following statements and tick them if they are true.

- ☒ In a sequent, premises entail a conclusion.
- ☐ In a sequent, conclusions entail a premise.
- ☐ A sequent is valid, if no proof for it can be found.
- ☒ A sequent is valid, if a proof for it can be found.

Proof Rules

Propositional Logic

	Introduction	Elimination
$\wedge$	$\dfrac{\varphi \quad \psi}{\varphi \wedge \psi}\ \wedge\text{i}$	$\dfrac{\varphi \wedge \psi}{\varphi}\ \wedge\text{e1}$ $\quad \dfrac{\varphi \wedge \psi}{\psi}\ \wedge\text{e2}$
$\vee$	$\dfrac{\varphi}{\varphi \vee \psi}\ \vee\text{i1}$ $\quad \dfrac{\varphi}{\psi \vee \varphi}\ \vee\text{i2}$	$\dfrac{\varphi \vee \psi \quad \boxed{\begin{array}{l}\varphi \text{ ass.}\\ \vdots \\ \chi\end{array}} \quad \boxed{\begin{array}{l}\psi \text{ ass.}\\ \vdots \\ \chi\end{array}}}{\chi}\ \vee\text{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{l}\varphi \text{ ass.}\\ \vdots \\ \psi\end{array}}}{\varphi \rightarrow \psi}\ \rightarrow\text{i}$	$\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}\ \rightarrow\text{e}$
$\neg$	$\dfrac{\boxed{\begin{array}{l}\varphi \text{ ass.}\\ \vdots \\ \bot\end{array}}}{\neg\varphi}\ \neg\text{i}$	$\dfrac{\varphi \quad \neg\varphi}{\bot}\ \neg\text{e}$
$\bot$	no rule	$\dfrac{\bot}{\varphi}\ \bot\text{e}$
$\neg\neg$	$\dfrac{\varphi}{\neg\neg\varphi}\ \neg\neg\text{i}$	$\dfrac{\neg\neg\varphi}{\varphi}\ \neg\neg\text{e}$

Derived Rules

$$\dfrac{}{\varphi \vee \neg\varphi}\ \text{LEM} \qquad \dfrac{\boxed{\begin{array}{l}\neg\varphi \text{ ass.}\\ \vdots \\ \bot\end{array}}}{\varphi}\ \text{PBC} \qquad \dfrac{\varphi \rightarrow \psi \quad \neg\psi}{\neg\varphi}\ \text{MT}$$

https://teaching.iaik.tugraz.at/_media/lub/deduction.pdf

Rules for Conjunction

$$\frac{\varphi \qquad \psi}{\varphi \land \psi} \land i$$

$$\frac{\varphi \land \psi}{\phi} \land e_1 \qquad \frac{\varphi \land \psi}{\psi} \land e_2$$

Proofs

$$\frac{\varphi \qquad \psi}{\varphi \land \psi} \land i$$

$$p, q \vdash p \land q$$

1.	p	prem.
2.	q	prem.
3.	$p \land q$	$\land$i 1,2

Example

$$p \wedge q, r \ \vdash \ q \wedge r$$

1.	$p \wedge q$	prem.
2.	r	prem.
3.	q	$\wedge$e2 1
4.	$q \wedge r$	$\wedge$i 3,2

$$\frac{\varphi \wedge \psi}{\phi} \wedge e_1$$

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi} \wedge i$$

Example

$$(p \wedge q) \wedge r, s \wedge t \vdash q \wedge s$$

1.	$(p \wedge q) \wedge r$	prem.
2.	$s \wedge t$	prem.
3.	$p \wedge q$	$\wedge$e1 1
4.	q	$\wedge$e1 2
5.	s	$\wedge$e1 2
6.	$q \wedge s$	$\wedge$i 4,5

Rules for Double Negation

Elimination

$$\frac{\neg\neg\varphi}{\varphi}\ \neg\neg\text{e}$$

Introduction

$$\frac{\varphi}{\neg\neg\varphi}\ \neg\neg i$$

Example 3

[Lecture] $\neg\neg\neg p \wedge q, \neg\neg r \vdash r \wedge \neg p \wedge \neg\neg q$

1.	$\neg\neg\neg p \wedge q$	prem.
2.	$\neg\neg r$	prem.
3.		
4.		
5.		
6.		
7.		
8.		
9.	$r \wedge \neg p \wedge \neg\neg q$	

Rules for Implication - Elimination

Elimination

$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \to\mathrm{e}$$

Derived Elimination Rule -
Modus Tollens

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi} \mathrm{MT}$$

Example 4

[Lecture] $p, p \to q, p \to (q \to r) \vdash r$

$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \to e$$

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi} \text{MT}$$

1.	p	prem.
2.	$p \to q$	prem.
3.	$p \to (q \to r)$	prem.
4.	$q \to r$	$\to$e 1,3
5.	q	$\to$e 1,2
6.	r	$\to$e 4,5

Example 5

$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \to e$$

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi} \text{MT}$$

[Lecture] $\neg p \to q, \neg\neg\neg q \land r \vdash p \land \neg\neg\neg q$

1.	$\neg p \to q$	prem.
2.	$\neg\neg\neg q \land r$	prem.
3.	$\neg\neg\neg q$	$\land$e1 2
4.	$\neg q$	$\neg\neg$e 3
5.	$\neg\neg p$	MT 1,4
6.	p	$\neg\neg$e 5
7.	$p \land \neg\neg\neg q$	$\land$i 6,3

Example 6

$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \to e$$

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi} \text{MT}$$

[Lecture] $\neg p \to (q \to r), \neg p, \neg r \vdash \neg q$

1.	$\neg p \to (q \to r)$	prem.
2.	$\neg p$	prem.
3.	$\neg r$	prem.
4.	$q \to r$	$\to$e 1,2
5.	$\neg q$	MT 4,3

Rules for Implication

Elimination

$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \to\mathrm{e}$$

Derived Elimination Rule - Modus Tollens

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi} \mathrm{MT}$$

Introduction

$$\frac{\boxed{\begin{array}{l}\varphi \ \text{ass.} \\ \vdots \\ \psi\end{array}}}{\varphi \to \psi} \to\mathrm{i}$$

Example 7

[Lecture] $p \to q, q \to r \vdash p \to r$

1.	$p \to q$	prem.
2.	$q \to r$	prem.
3.	p	ass.
4.	q	$\to$e 3,1
5.	r	$\to$e 4,2
6.	$p \to r$	$\to$i 3-5

$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \to\text{e}$$

$$\frac{\boxed{\begin{array}{l}\varphi \text{ ass.} \\ \vdots \\ \psi\end{array}}}{\varphi \to \psi} \to\text{i}$$

Example 8

$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \to\mathrm{e}$$

[Lecture] $p \to (q \wedge r), (q \to s) \vdash p \to (s \wedge r)$

1.	$p \to (q \wedge r)$	prem.
2.	$q \to s$	prem.
3.	p	ass.
4.	$q \wedge r$	$\to$e 1,3
5.	q	$\wedge$e1 4
6.	s	$\to$e 2,5
7.	r	$\wedge$e2 4
8.	$s \wedge r$	$\wedge$i 6,7
9.	$p \to (s \wedge r)$	$\to$i 3-8

$$\frac{\boxed{\begin{array}{l}\varphi \ \text{ass.} \\ \vdots \\ \psi\end{array}}}{\varphi \to \psi} \to\mathrm{i}$$

Rules for Disjunction - Introduction

Introduction

$$\frac{\varphi}{\varphi \vee \psi} \vee\mathrm{i1} \qquad \frac{\varphi}{\psi \vee \varphi} \vee\mathrm{i2}$$

Example 9

[Lecture] $p \wedge q, r \rightarrow s \ \vdash \ (p \vee (r \rightarrow s)) \wedge (q \vee ((t \vee r) \rightarrow u))$

1.	$p \wedge q$	prem.
2.	$r \rightarrow s$	prem.
3.	p	$\wedge$e1 1
4.	$p \vee (r \rightarrow s)$	$\vee$i1 3
5.	q	$\wedge$e2 1
6.	$q \vee ((t \vee r) \rightarrow u)$	$\vee$i1 5
7.	$(p \vee (r \rightarrow s)) \wedge (q \vee ((t \vee r) \rightarrow u))$	$\wedge$i 4,6

Rules for Disjunction

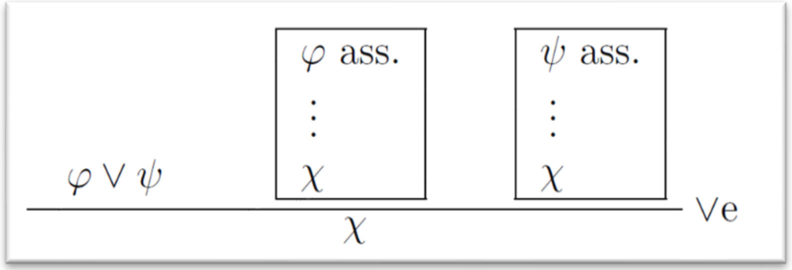
Elimination

$$\dfrac{\varphi \vee \psi \qquad \boxed{\begin{array}{l} \varphi \ \text{ass.} \\ \vdots \\ \chi \end{array}} \qquad \boxed{\begin{array}{l} \psi \ \text{ass.} \\ \vdots \\ \chi \end{array}}}{\chi} \ \vee\text{e}$$

Introduction

$$\dfrac{\varphi}{\varphi \vee \psi} \ \vee\text{i1} \qquad \dfrac{\varphi}{\psi \vee \varphi} \ \vee\text{i2}$$

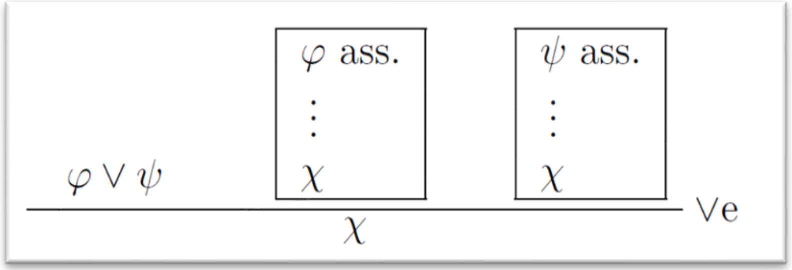
Example



$$p \vee q \;\vdash\; q \vee p$$

1.	$p \vee q$	prem.
2.		
3.		
4.		
5.		
6.	$q \vee p$	

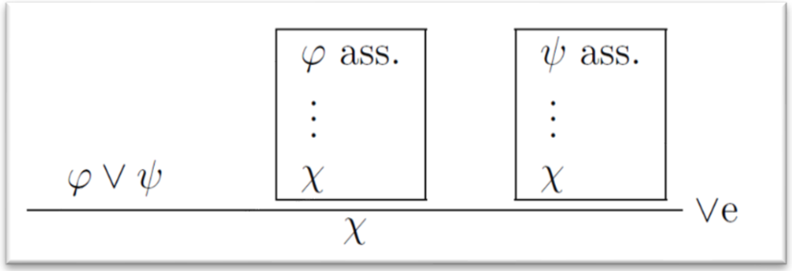
Example



$$p \vee q \ \vdash \ q \vee p$$

1.	$p \vee q$	prem.
2.	p	ass.
3.	$q \vee p$	∨i2 2
4.	q	ass.
5.	$q \vee p$	∨i1 4
6.	$q \vee p$	∨e 1,2-3,4-5

Example 10



[Lecture] $q \rightarrow r \vdash \; p \vee q \rightarrow p \vee r$

1.	$q \rightarrow r$	prem.
2.	$p \vee q$	ass.
3.	p	ass.
4.	$p \vee r$	∨i1 2
5.	q	ass.
6.	r	→e 5,1
7.	$p \vee r$	∨i 6
8.	$p \vee r$	∨e 2,3-4,5-7
9.	$p \vee q \rightarrow (p \vee r)$	→i 2-8

Rules for Negation

Elimination

$$\frac{\varphi \qquad \neg\varphi}{\bot}\ \neg\mathrm{e}$$

Introduction

$$\frac{\boxed{\begin{array}{l}\varphi \ \text{ass.}\\ \vdots \\ \bot\end{array}}}{\neg\varphi}\ \neg\mathrm{i}$$

Rule for ⊥ - Elimination

$$\frac{\bot}{\varphi}\ \bot\mathrm{e}$$

Derived Rule - Proof by Contradiction

$$\frac{\boxed{\begin{array}{l}\neg\varphi \ \text{ass.}\\ \vdots \\ \bot\end{array}}}{\varphi}\ \mathrm{PBC}$$

Other Rules

Law-of-the-Excluded-Middle Rule

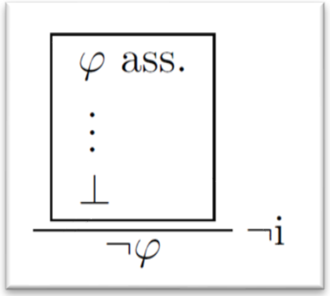
$$\frac{}{\varphi \vee \neg\varphi}\ \text{LEM}$$

Copy-Rule

$$\frac{\varphi}{\varphi}\ \text{copy}$$

Example 11

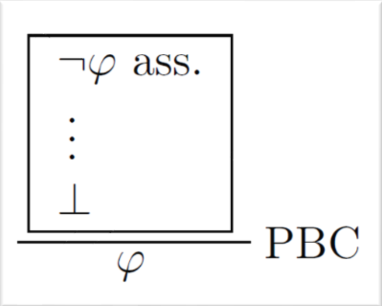
[Lecture] $p \to \neg q, q \vdash \neg p$



1.	$p \to \neg q$	prem.
2.	q	prem.
3.	p	ass.
4.	$\neg q$	$\to$e 3,1
5.	$\bot$	$\neg$e 2,4
6.	$\neg p$	$\neg$i 3-5

Example 12

[Lecture] $\neg p \rightarrow \neg q, q \vdash p$



1.	$\neg p \rightarrow \neg q$	prem.
2.	q	prem.
3.	$\neg p$	ass.
4.	$\neg q$	$\rightarrow$e 3,1
5.	$\bot$	$\neg$e 2,4
6.	p	PBC 3-5

Example 13

[Lecture] $\vdash p \to (q \to p)$

1.	p	ass.
2.	q	ass.
3.	p	copy 1
4.	$q \to p$	$\to$i 2-3
5.	$p \to q \to p$	$\to$i 1-4

Example 14

[Lecture] $\vdash ((p \to q) \to p) \to p$

1.	$(p \to q) \to p$	ass.
2.	$\neg p$	ass.
3.	$\neg(p \to q)$	MT 1,2
4.	p	ass.
5.	$\bot$	$\neg$e 2,4
6.	q	$\bot$e 5
7.	$p \to q$	$\to$i 4-6
8.	$\bot$	$\neg$e 3,7
9.	p	PBC 2-8
10.	$((p \to q) \to p) \to p$	$\to$i 1-9

Example 15

[Lecture] $\neg q \vee \neg p \;\vdash\; \neg(q \wedge p)$

1.	$\neg q \vee \neg p$	prem.
2.	$q \wedge p$	ass.
3.	$\neg q$	ass.
4.	q	$\wedge$e1 2
5.	$\bot$	$\neg$e 3,4
6.	$\neg p$	ass.
7.	p	$\wedge$e2 2
8.	$\bot$	$\neg$e 6,7
9.	$\bot$	$\vee$e 1, 3-5, 6-8
10.	$\neg(q \wedge p)$	$\neg$i 2-9

Example 16

[Lecture] $p \vee \neg\neg q, \neg p \wedge \neg q \vdash s \vee \neg t$

1.	$p \vee \neg\neg q$	prem.
2.	$\neg p \wedge \neg q$	prem.
3.	p	ass.
4.	$\neg p$	$\wedge$e1 2
5.	$\bot$	$\neg$e 3,4
6.	$s \vee \neg t$	$\bot$e 5
7.	$\neg\neg q$	ass.
8.	$\neg q$	$\wedge$e2 2
9.	$\bot$	$\neg$e 7,8
10.	$s \vee \neg t$	$\bot$e 9
11.	$s \vee \neg t$	$\vee$e 1, 3-6, 7-10

Example 17

[Lecture] $p \to q \;\vdash\; \neg p \vee q$

1.	$p \to q$	prem.
2.	$\neg p \vee p$	LEM
3.	$\neg p$	ass.
4.	$\neg p \vee q$	$\vee$i1 3
5.	p	ass.
6.	q	$\to$e 1,5
7.	$\neg p \vee q$	$\vee$i2 6
8.	$\neg p \vee q$	$\vee$e 2,3-5,5-7

Example 18

[Lecture] $\vdash \neg(p \wedge q) \vee p$

1.	$p \vee \neg p$	LEM
2.	p	ass.
3.	$\neg(p \wedge q) \vee p$	$\vee$i2 2
4.	$\neg p$	ass.
5.	$p \wedge q$	ass.
6.	p	$\wedge$e1 5
7.	$\bot$	$\neg$e 6,4
8.	$\neg(p \wedge q)$	$\neg$i 5-7
9.	$\neg(p \wedge q) \vee p$	$\vee$i1 8
10.	$\neg(p \wedge q) \vee p$	$\vee$e 1, 2-3, 4-9

Example 19

[Lecture] $\neg\neg k \to (l \vee m), \neg\neg\neg l \to m \;\vdash\; \neg k \vee (l \vee \neg\neg m)$

1.	$\neg\neg k \to (l \wedge m)$	prem.
2.	$\neg\neg\neg l \to m$	prem.
3.	$m \vee \neg m$	LEM
4.	m	ass.
5.	$\neg\neg m$	$\neg\neg$i 4
6.	$l \vee \neg\neg m$	$\vee$i2 5
7.	$\neg k \vee (l \vee \neg\neg m)$	$\vee$i2 6
8.	$\neg m$	ass.
9.	$\neg\neg\neg\neg l$	MT 2, 8
10.	$\neg\neg l$	$\neg\neg$e 9
11.	l	$\neg\neg$e 10
12.	$l \vee \neg\neg m$	$\vee$i1 11
13.	$\neg k \vee (l \vee \neg\neg m)$	$\vee$i2 12
14.	$\neg k \vee (l \vee \neg\neg m)$	$\vee$e 3, 4-7, 8-13

Tips for Deduction

- Work from both sides
- Look at the conclusion
  - If it is of the form $\varphi \to \psi$, apply immediately $\to i$
  - If it is of the form $\neg\varphi$, apply immediately $\neg i$
- If you get stuck
  - Try case splits: LEM
  - Try proof by contradiction

Outline

- Proof Rules ✔
  - Introduction Rules
  - Elimination Rules
- Soundness and Completeness
  - Proof the invalidity of sequences via counter examples

Soundness ("Korrektheit")

- Definition

$$\boldsymbol{\phi_1, \phi_2, \dots, \phi_n \vdash \psi \qquad \Rightarrow \qquad \phi_1, \phi_2, \dots, \phi_n \vDash \psi}$$

- Meaning
  - Every provable sequent is a correct semantic entailment.
  - Incorrect entailments are not provable.
    - $\phi_1, \phi_2, \dots, \phi_n \nvDash \psi \qquad \Rightarrow \qquad \phi_1, \phi_2, \dots, \phi_n \nvdash \psi$

Completeness ("Vollständigkeit")

- Definition

$$\phi_1, \phi_2, \dots, \phi_n \vDash \psi \quad \Rightarrow \quad \phi_1, \phi_2, \dots, \phi_n \vdash \psi$$

- Meaning
  - Every correct semantic entailment has a proof.
  - Unprovable sequents are incorrect entailments.
    - $\phi_1, \phi_2, \dots, \phi_n \nvdash \psi \quad \Rightarrow \quad \phi_1, \phi_2, \dots, \phi_n \nvDash \psi$

Example 20

[Lecture] "Natural deduction for propositional logic is *sound* and *complete*." Explain in your own words what this means.

Soundness

Natural deduction for propositional logic is sound. Therefore, any sequent that can be proven is a correct semantic entailment.

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi \quad \Rightarrow \quad \phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

So, if we have proven with natural deduction that a sequent $\phi_1, \phi_2, \ldots, \phi_n$ is valid, then for all valuations in which all premises $\phi_1, \phi_2, \ldots, \phi_n$ evaluate to *true*, ψ evaluates to *true* as well.

Completeness

Natural deduction for propositional logic is sound. Therefore, any sequent that is a correct semantic entailment can be proven.

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi \quad \Rightarrow \quad \phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

Invalid Sequents

- $p \vee q \vdash p \wedge q$?

- Model $\mathcal{M}$: $p = T$ $q = F$
  - $\mathcal{M} \models \mathrm{p} \vee q$ but $\mathcal{M} \not\models \mathrm{p} \wedge q$

  - $\mathcal{M}$ satisfies all premises
  - $\mathcal{M}$ does not satisfy the conclusion
  - *Therefore, $\mathcal{M}$ is a counterexample!*

  - This proves: $\mathrm{p} \vee q \nvdash \mathrm{p} \wedge q$

Example 21

[Lecture] How can you show that a sequent is not valid? Is this a consequence of soundness or completeness. Explain your answer.

To show that a sequent is invalid, we need to find a *counter example*. A counter example is a model, that *satisfies all premises but falsifies the conclusion.*

Example 22

[Lecture] $p \to q, q \to r \ \vdash \ r$.

This sequent is not provable.

$\mathcal{M} : p = F, q = F, r = F$

$\mathcal{M} \models p \to q, q \to r$

$\mathcal{M} \not\models r$

Example 23

[Lecture] Translate the following reasoning into a sequent. If the sequent is valid, proof it using the rules of natural deduction. If the sequent is not valid, provide a counter example.

If I press the button, the window opens.
The window is open.
Therefore, I pressed the button.

Example 23

[Lecture] Translate the following reasoning into a sequent. If the sequent is valid, proof it using the rules of natural deduction. If the sequent is not valid, provide a counter example.

Translation:

p : Press button.

q : Open window.

If I press the button, the window opens.	$p \to q$
The window is open.	q
Therefore, I pressed the button.	$\vdash p$

sequent: $p \to q, q \vdash p$

This sequent is not provable.

$\mathcal{M} : p = F, q = T$

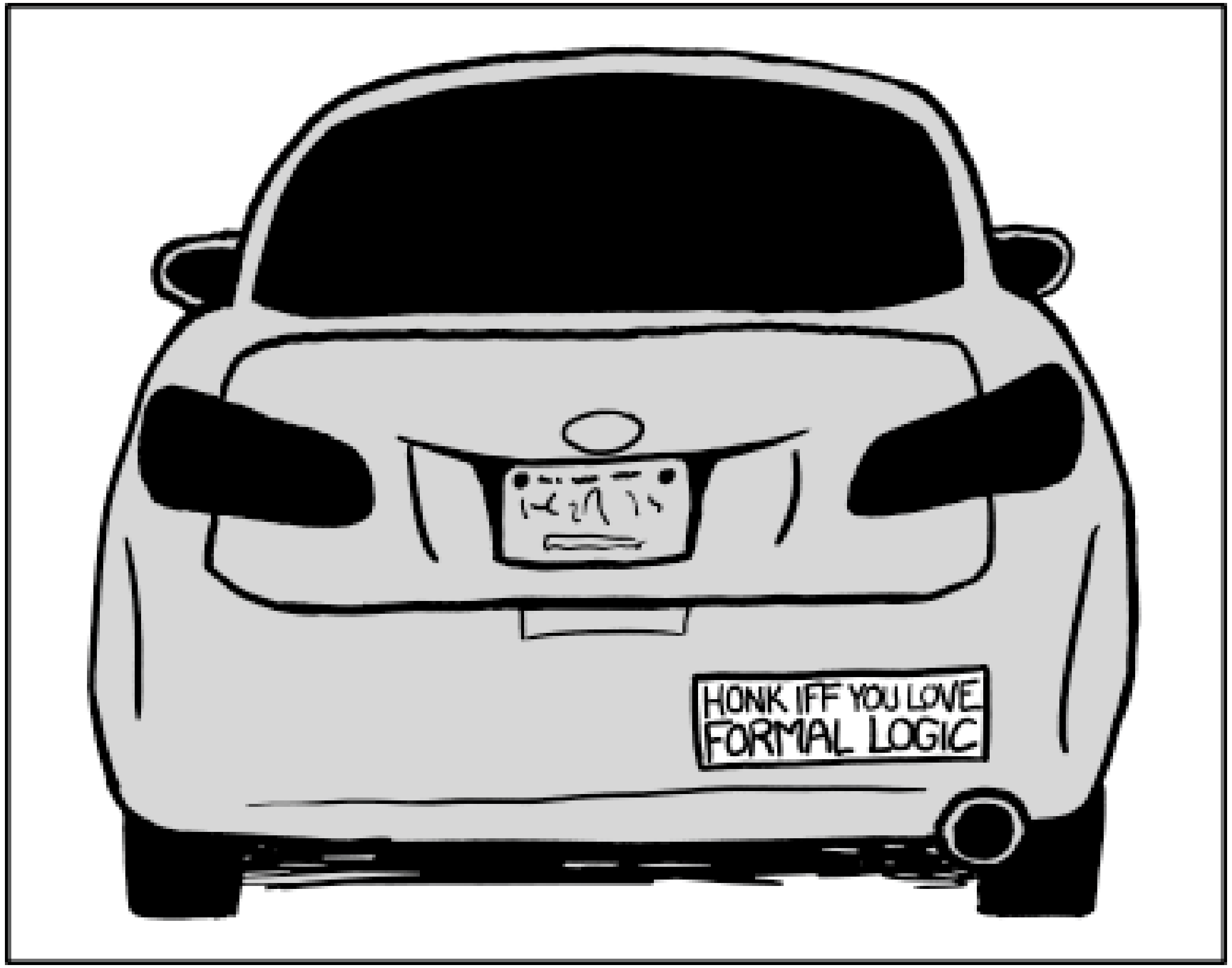
$\mathcal{M} \models p \to q, q$

$\mathcal{M} \not\models p$

Learning Targets

- Perform deduction proofs or find a counterexample for sequents
- Check or find errors in a given deduction proof
- Explain “soundness” and “completeness”
  - Of natural deduction for propositional logic

Thank You



https://xkcd.com/1033/