Logic and Computability

Lecture 7

Natural Deduction for Predicate Logic - Part 2

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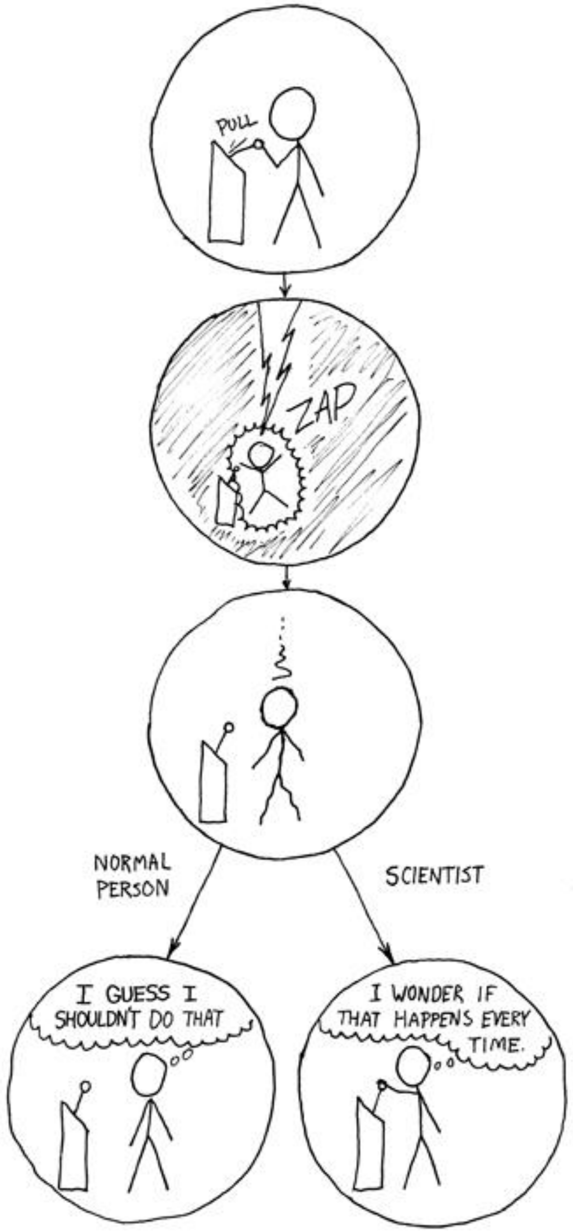

https://xkcd.com/2497/

Proof Rules for Existential Quantification

$$\frac{\Phi\,[t/x]}{\exists x\ \Phi}\ \exists\mathrm{i}$$

$$\frac{\exists x\ \Phi \qquad \boxed{\begin{array}{l} x_0 \\ \qquad \Phi\,[x_0/x]\ \text{ass.} \\ \qquad \vdots \\ \qquad \chi \end{array}}\ \ x_0\ \text{fresh}}{\chi}\ \exists\mathrm{e}$$

Examples for Proofs with Existential Quantification

- 5 [Lecture] $\forall x \big(P(x) \rightarrow Q(y) \big), \ \forall y \big(P(y) \land R(x) \big) \ \vdash \ \exists x \, Q(x)$

Examples for Proofs with Existential Quantification

- 5 [Lecture] $\forall x\big(P(x) \to Q(y)\big), \ \forall y\big(P(y) \wedge R(x)\big) \ \vdash \ \exists x\, Q(x)$

1.	$\forall x\ (P(x) \to Q(y))$	prem.
2.	$\forall y\ (P(y) \wedge R(x))$	prem.
3.	$P(t) \to Q(y)$	$\forall$e 1
4.	$P(t) \wedge R(x)$	$\forall$e 2
5.	$P(t)$	$\wedge e_1$ 4
6.	$Q(y)$	$\to$e 3
7.	$\exists x\ Q(x)$	$\exists$i 6

Examples for Proofs with Existential Quantification

- 6 [Lecture] $\exists x\big(P(x) \rightarrow Q(y)\big),\ \forall x P(x) \vdash Q(y)$

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- 6 [Lecture] $\exists x\big(P(x) \rightarrow Q(y)\big),\ \forall x P(x) \vdash Q(y)$

1.		$\exists x\ (P(x) \rightarrow Q(y))$	prem.
2.		$\forall x\ P(x)$	prem.
3.	x_0	$P(x_0) \rightarrow Q(y)$	ass.
4.		$P(x_0)$	$\forall$e 2
5.		$Q(y)$	$\rightarrow$e 3,4
6.		$Q(y)$	$\exists$e 3-5

Examples for Proofs with Existential Quantification

- 7 [Lecture] $\exists x \neg P(x), \quad \forall x \neg Q(x) \vdash \quad \exists x\, (\neg P(x) \land \neg Q(x))$

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- 7 [Lecture] $\exists x \neg P(x), \quad \forall x \neg Q(x) \vdash \quad \exists x\, (\neg P(x) \land \neg Q(x))$

1.		$\exists x\ \neg P(x)$	prem.
2.		$\forall x\ \neg Q(x)$	prem.
3.	x_0	$\neg P(x_0)$	ass.
4.		$\neg Q(x_0)$	$\forall$e 2
5.		$\neg P(x_0) \land \neg Q(x_0)$	$\land$i 3,4
6.		$\exists x\ (\neg P(x) \land \neg Q(x))$	$\exists$i 5
7.		$\exists x\ (\neg P(x) \land \neg Q(x))$	$\exists$e 3-6

Counterexamples

- 8 [Lecture] $\exists x\big(P(x) \rightarrow Q(y)\big), \exists x P(x) \quad \vdash \quad Q(y)$

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This sequent is not provable.
Model $\mathcal{M}$:

$$\mathcal{A} = \{a, b\}$$
$$P^{\mathcal{M}} = \{a\}$$
$$Q^{\mathcal{M}} = \{a\}$$
$$y \leftarrow b$$

$\mathcal{M} \models \exists x\ (P(x) \to Q(y)), \quad \exists x\ P(x)$
$\mathcal{M} \not\models Q(y)$

Examples

- 9 [Lecture] Consider the following natural deduction proof for the sequent

$$\forall x\ (P(x) \to Q(x)),\quad \exists x\ P(x) \quad\vdash\quad \forall x Q(x).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.

1.	$\forall x\ (P(x) \to Q(x))$	prem.
2.	$\exists x\ P(x)$	prem.
3.	x_0	
4.	$P(x_0)$	ass.
5.	$P(x_0) \to Q(x_0)$	$\forall$e 1
6.	$Q(x_0)$	$\to$ e, 4,5
7.	$\forall x\ Q(x)$	$\forall$i 4-6
8.	$\forall x\ Q(x)$	$\exists$e 2,3-7

Examples

- 9 [Lecture] Consider the following natural deduction proof for the sequent

$$\forall x\ (P(x) \to Q(x)),\quad \exists x\ P(x) \quad \vdash \quad \forall x Q(x).$$

This sequent is not provable.
Model $\mathcal{M}$:

$$\mathcal{A} = \{a, b\}$$
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$\mathcal{M} \models \forall x\ (P(x) \to Q(x)),\quad \exists x\ P(x)$
$\mathcal{M} \not\models \forall x Q(x)$

Examples for Proofs with Existential Quantification

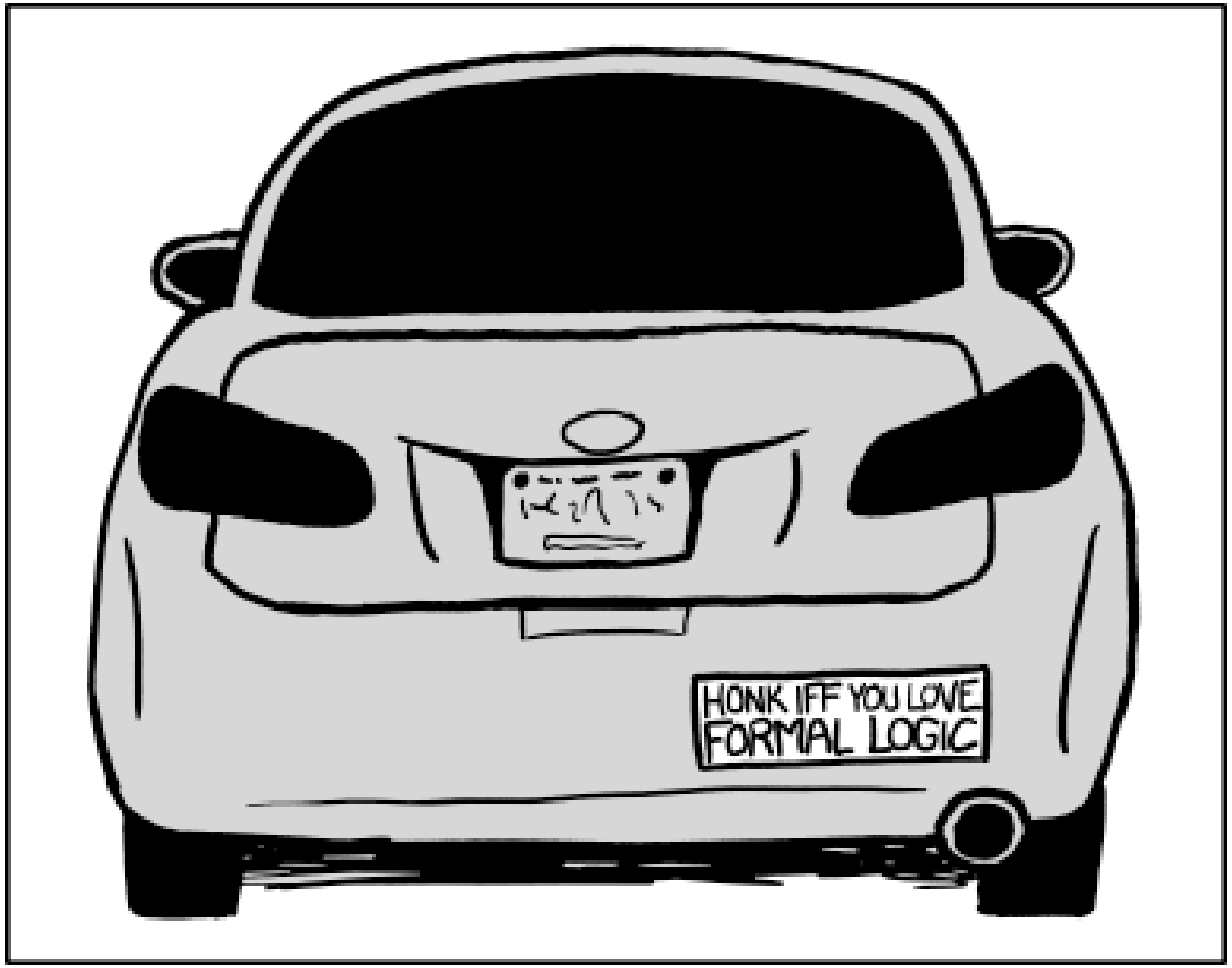
- 10 [Lecture] $\forall x \neg \big(P(x) \wedge Q(x)\big) \vdash \neg \exists x \, (P(x) \wedge Q(x))$

Examples for Proofs with Existential Quantification

- 10 [Lecture] $\forall x \neg\big(P(x) \land Q(x)\big) \vdash \neg\exists x\,(P(x) \land Q(x))$

1.		$\forall x\, \neg(P(x) \land Q(x))$	prem.
2.		$\exists x\, (P(x) \land Q(x))$	ass.
3.	t	$P(t) \land Q(t)$	ass.
4.		$\neg P(t) \land Q(t)$	$\forall$e 1
5.		$\bot$	$\neg$e 3,4
6.		$\bot$	$\exists$e 3-5
7.		$\neg\exists x\, (P(x) \land Q(x))$	$\neg$i 2-6

Thank You



https://xkcd.com/1033/