SCIENCE
PASSION
TECHNOLOGY

Logic and Computability

Lecture 6

Natural Deduction for Predicate Logic

PART 1

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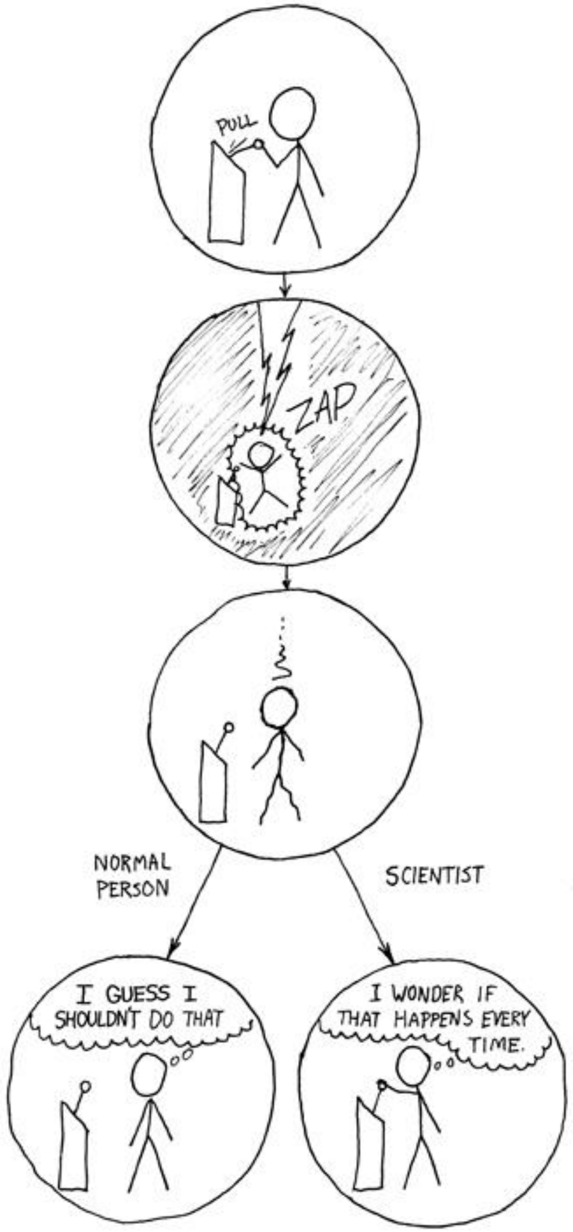


https://xkcd.com/2497/

Motivation

- Extend Natural Deduction to Predicate Logic
  - Richer Language → More powerful proofs
- Basis for “real proofs”

Outline

- New Rules for Natural Deduction
  - Quantifiers
  - Equality
- Many examples

Proof Rules for Universal Quantification

$$\frac{\forall x \; \Phi}{\Phi \, [t/x]} \; \forall \mathrm{e}$$

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad \vdots \\ \quad \Phi \, [x_0/x] \end{array}} \quad x_0 \text{ fresh}}{\forall x \; \Phi} \; \forall \mathrm{i}$$

Examples for Proofs with Universal Quantification

$$\frac{\forall x\ \Phi}{\Phi\,[t/x]}\ \forall\mathrm{e}$$

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad\Phi\,[x_0/x] \end{array}}\quad x_0 \text{ fresh}}{\forall x\ \Phi}\ \forall\mathrm{i}$$

Examples for Proofs with Universal Quantification

- 1 [Lecture] $\forall x(P(x) \rightarrow Q(x)), \forall x P(x) \vdash \forall x Q(x)$

1.		$\forall x\ (P(x) \rightarrow Q(x))$	prem.
2.		$\forall x\ P(x)$	prem.
3.	x_0	$P(x_0) \rightarrow Q(x_0)$	$\forall$e 1
4.		$P(x_0)$	$\forall$e 2
5.		$Q(x_0)$	$\rightarrow$e 3,4
6.		$\forall x\ Q(x)$	$\forall$i 3-5

Examples for Proofs with Universal Quantification

- 2 [Lecture] $\forall x P(x) \land \forall x (P(y) \to Q(x)) \vdash \quad Q(z)$

1.	$\forall x\ P(x) \land \forall x\ (P(y) \to Q(x))$	prem.
2.	$\forall x\ P(x)$	$\land e_1$ 1
3.	$\forall x\ (P(y) \to Q(x))$	$\land e_2$ 1
4.	$P(y)$	$\forall e$ 2
5.	$P(y) \to Q(z)$	$\forall e$ 3
6.	$Q(z)$	$\to e$ 5,4

Examples for Proofs with Universal Quantification

- 3 [Lecture] $\forall x\big(P(x) \wedge Q(x)\big) \vdash \forall x\, P(x) \wedge \forall x Q(x)$

1.		$\forall x\ (P(x) \wedge Q(x))$	prem.
2.	x_0	$P(x_0) \wedge Q(x_0)$	$\forall$e 1
3.		$P(x_0)$	$\wedge$e 2
4.		$\forall x\ P(x)$	$\forall$i2 – 3
5.	x_1	$P(x_1) \wedge Q(x_1)$	$\forall$e 1
6.		$Q(x_1)$	$\wedge$e 5
7.		$\forall x\ Q(x)$	$\forall$i5 – 6
8.		$\forall x\ P(x) \wedge \forall x\ Q(x)$	$\wedge$i 4,7

Examples for Proofs with Universal Quantification

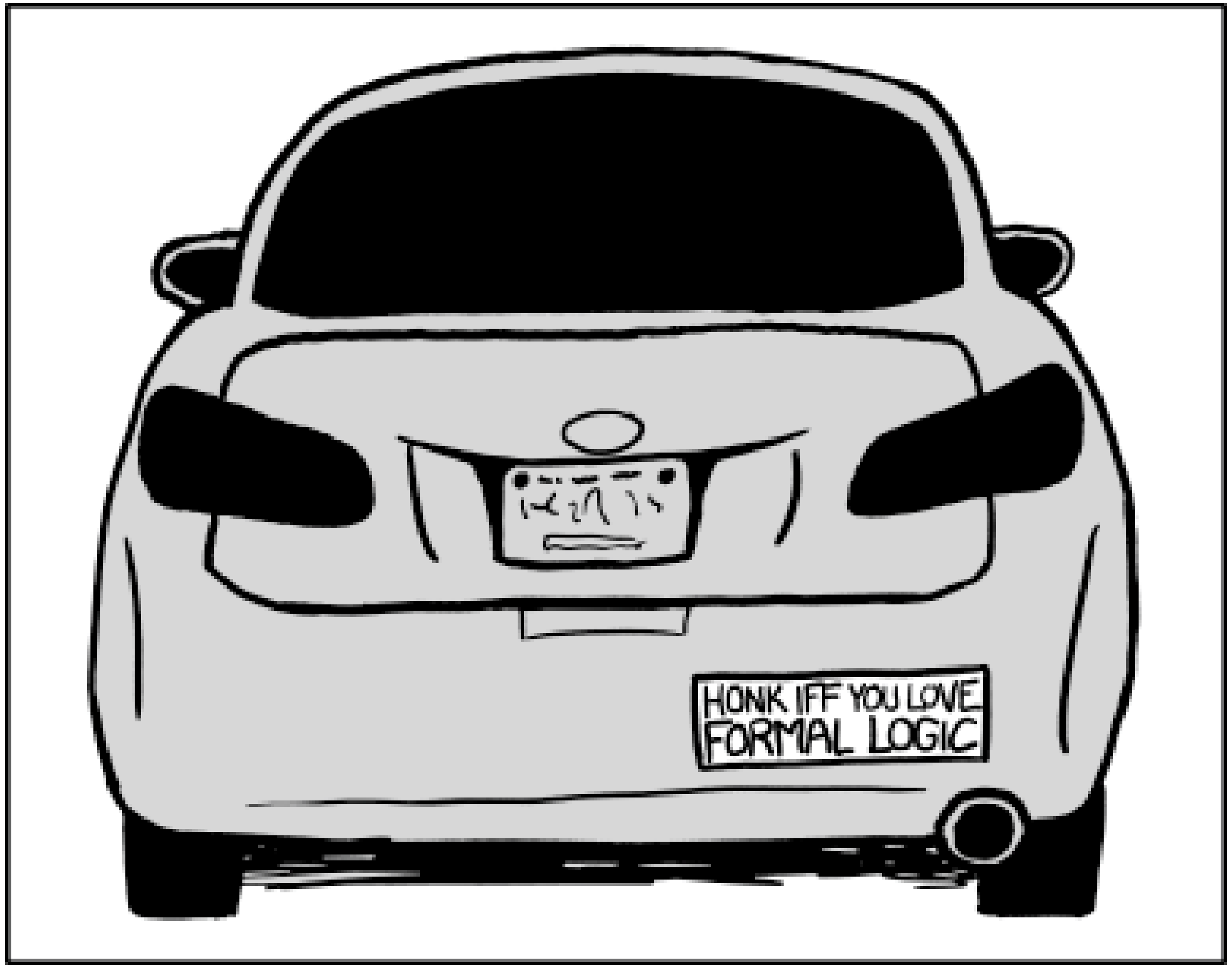
- 4 [Lecture] $\forall x P(x) \vee \forall x\, Q(x) \;\vdash\; \forall y(\,P(y) \vee Q(y))$

1.		$\forall x\; P(x) \vee \forall x\; Q(x)$	prem.
2.		$\forall x\; P(x)$	ass.
3.	t	$P(t)$	$\forall$e 2
4.		$P(t) \vee Q(t)$	$\vee \mathrm{i}_1$ 3
5.		$\forall y\; (P(y) \vee Q(y))$	$\forall$i 3-4
6.		$\forall x\; Q(x)$	ass.
7.	s	$Q(s)$	$\forall$e 6
8.		$P(s) \vee Q(s)$	$\vee \mathrm{i}_2$ 7
9.		$\forall y\; (P(y) \vee Q(y))$	$\forall$i 7-8
10.		$\forall y\; (P(y) \vee Q(y))$	ve 1,2-5,6-9

Next Time

- Rules for existential quantification
- More complex examples
- Decidability of natural deduction for predicate logic

Thank You



https://xkcd.com/1033/