TU Graz

SCIENCE
PASSION
TECHNOLOGY

Logic and Computability

Lecture 5

Predicate Logic

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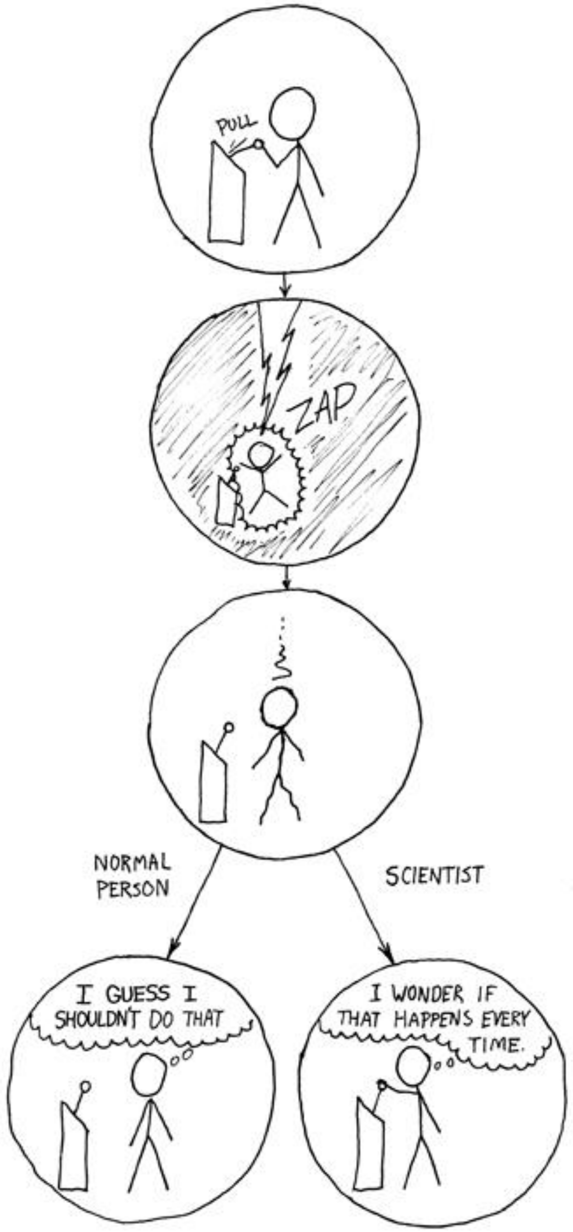

https://xkcd.com/2497/

Outline

- Modelling Sentences
- Syntax
- Semantics, Models
  - Models
  - Satisfiability & Validity

Formulas in Predicate Logic

- Variables over arbitrary **domains**
- **Functions and predicates**
- New operators:
  - $\forall$ … $forall$
  - $\exists$ … $exists$

$$\forall x \exists y . P(x, f(y))$$

Examples of Formulas in Predicate Logic

- Some people in class visited the Grand Canyon

  - $A = \{\text{people}\}$ ... Domain of variables

  - Predicates:
    - InClass(x) ... Returns true if x is in class
    - VisitedGC(x) ... Returns true if x visted the Grand Canyon

  - $\exists \boldsymbol{x}.\,(\boldsymbol{InClass}\,(\boldsymbol{x}) \wedge \boldsymbol{VisitedGC}(\boldsymbol{x}))$

Examples of Formulas in Predicate Logic

- Not all birds can fly
  - $A = \{\text{birds}\}$
  - Predicates:
    - Fly(x) … Returns true if x can fly
  - $\neg\forall \boldsymbol{x}.\,(\boldsymbol{Fly}(\boldsymbol{x}))$

Examples of Formulas in Predicate Logic

- All integers are either even or odd
  - $A = \mathbb{N}$
  - Predicates:
    - Even(x) … Returns true if x is even
    - Odd(x) … Returns true if x is odd
  - $\forall \boldsymbol{x}.\,(\boldsymbol{Even}(\boldsymbol{x}) \oplus \boldsymbol{Odd}(\boldsymbol{x}))$

Examples of Formulas in Predicate Logic

1. [Lecture] Model the following declarative sentences with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

   (a) Alice has no sister.

   (b) A person who wears a crown is either a king or a queen.

   (c) Not everybody likes everybody.

   (d) Everybody loves somebody.

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(a)
- $A = \{\text{people}\}$
- Predicates:
  - $Alice(x)$ … x is Alice
  - $Sister(x)$ … x has a sister

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- $A = \{\text{people}\}$
- Predicates:
  - $Alice(x) \ldots x \text{ is } Alice$
  - $Sister(x) \ldots x \text{ has a sister}$
- $\forall x(Alice(x) \rightarrow \neg Sister(x))$

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(b)

- $A = \{\text{people}\}$
- $WearsCrown(x) \ldots x\ wears\ a\ crown$
- $King(x) \ldots x\ is\ a\ king$
- Queen(x) … x is a queen
- $\forall x(\boldsymbol{WearsCrown}(x) \rightarrow (\boldsymbol{King}(x) \vee \boldsymbol{Queen}(x))$

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(c)

- $A = \{\text{people}\}$
- $Likes(x, y)$ … x $likes$ y

- $\neg\forall x \forall y (\boldsymbol{Likes}(x, y))$

Examples of Formulas in Predicate Logic

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   (d) Everybody loves somebody.

(d)

- $A = \{\text{people}\}$
- $Loves(x, y) \ldots x\ loves\ y$

- $\forall x \exists y (Loves(x, y))$

Examples of Formulas in Predicate Logic

2. [Lecture] Consider the following declarative sentences:

- *"Every integer is greater or equal to one."*
- *"For any two integers, their sum is smaller than their product"*

Model this sentence with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

- $A = \mathbb{N}$
- x ≥ y … *x is greater or euqal to* 1 (in prefixed notation Geq(x,y))

(a)

- $\forall x (x \geq 1)$

Examples of Formulas in Predicate Logic

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Model this sentence with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

(b)

- $A = \mathbb{N}$
- $x + y$ returns the sum of x and y (Note: function, not a predicate)
- $x \cdot y$ returns the sum of x and y (Note: function, not a predicate)
- $x < y$ returns true if x is smaller than y

Examples of Formulas in Predicate Logic

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(b)

- $A = \mathbb{N}$
- x + y returns the sum of x and y (Note: function, not a predicate)
- x · y returns the sum of x and y (Note: function, not a predicate)
- x < y returns true if x is smaller than y

- $\forall x \forall y ((x + y) < (x \cdot y))$

Examples of Formulas in Predicate Logic

3. [Lecture] Consider the following declarative sentence (known as *Goldbach's Conjecture*):

"Every even integer greater than 2 is equal to the sum of two prime numbers."

Model this sentence with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

$\mathcal{A} = \mathbb{N}$
$E(x) \ldots x$ is even
$G(x) \ldots x$ is greater than 2
$P(x) \ldots x$ is prime

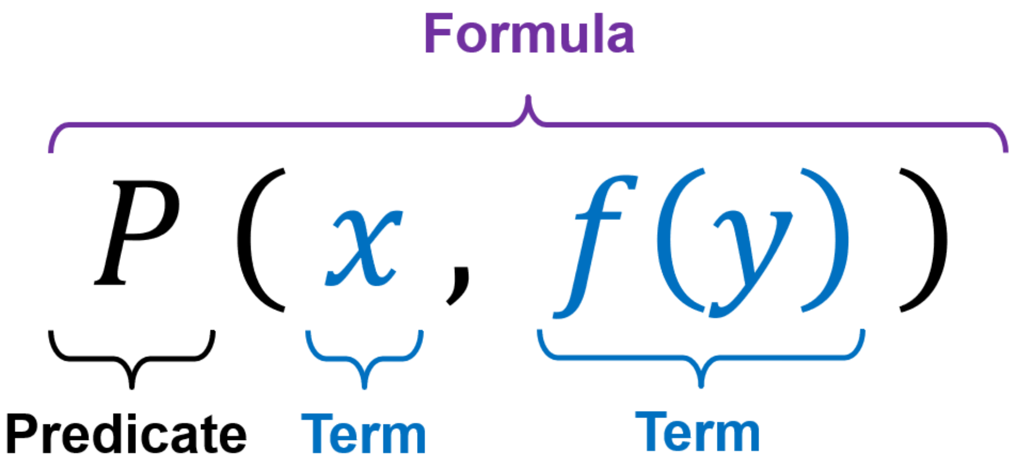
$$\forall x\,(E(x) \wedge G(x) \rightarrow \exists a, b(P(a) \wedge P(b) \wedge (x = a + b)))$$

Outline

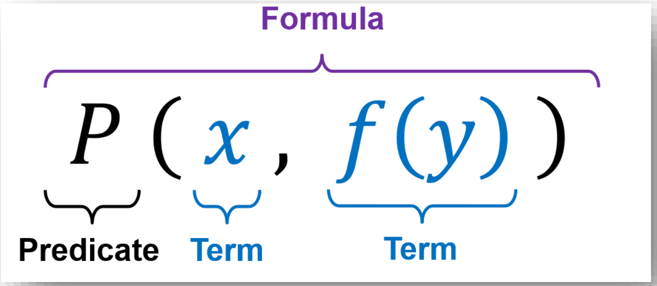
- Modelling Sentences ✔
- Syntax
- Semantics, Models
  - Models
  - Satisfiability & Validity

Syntax of Predicate Logic

- **Terms**
  - Refer to **Objects** of the domain:
    - *constants* represent individual objects, e.g., Alice, Bob, 5, 3…
    - *variables* like x, y represent objects
    - *functions symbols* refer to objects like $x \cdot y, f(x)$ …
- **Formulas**
  - Have a truth value: *predicates*
  - E.g., $x \cdot y == 1$



Syntax of Predicate Logic - Symbols

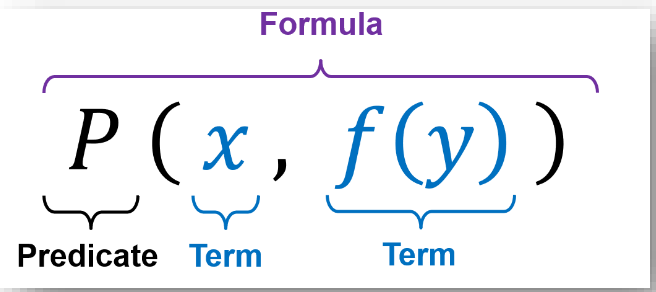


- Variables $\mathbb{V}$
  - E.g., $x, y, z, \ldots$

- Functions $\mathbb{F}$
  - $f, g, h, \ldots$ (arity > 0)
  - constants (arity = 0)

- Predicates $\mathbb{P}$
  - $P, Q, R, \ldots$ (arity > 0)
  - Prop. constants (arity = 0)

Syntax of Predicate Logic - Terms

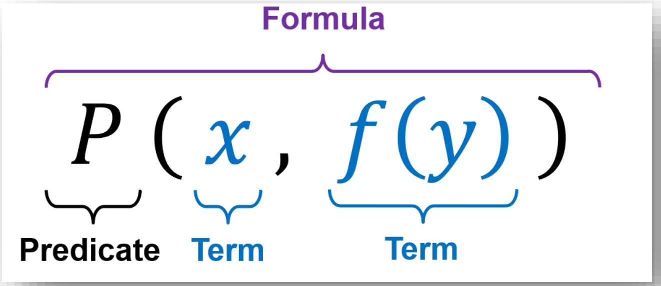


- Recursive Definition
  - Variable
  - Nullary Function (constant)
  - Given terms $t_1, t_2, \dots, t_n$, and n-ary Function f

$$\Updownarrow$$

$f(t_1, t_2, \dots, t_n)$ is a term

Syntax of Predicate Logic - Formulas

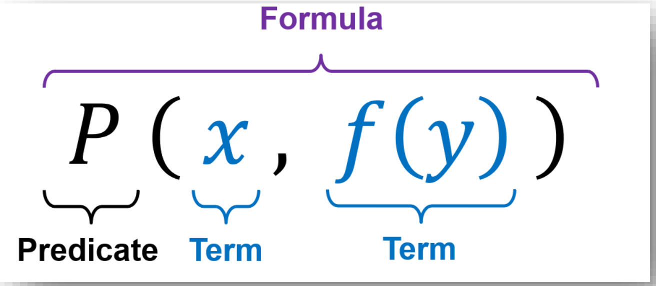


Preconditions:

- Terms $t_1, t_2, \dots, t_n$
- n-ary predicate symbol P
- formulas ϕ, ψ
- Variable x

- $P(t_1, t_2, \dots, t_n)$
- $\neg\phi$
- $\phi \land \psi$, $\phi \lor \psi$, $\phi \rightarrow \psi$
- $\forall x\, \phi$
- $\exists x\, \phi$

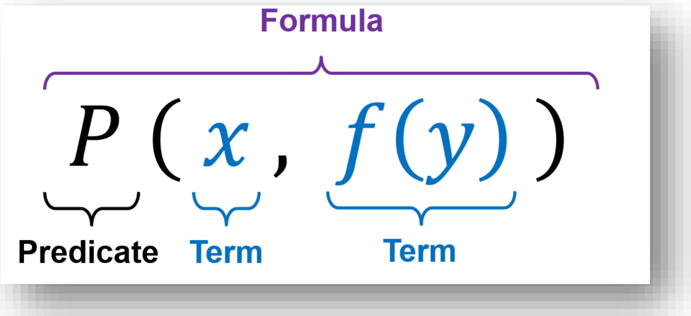
Binding Priorities



1. ∀, ∃, ¬
2. ∧
3. ∨
4. →
   - Right-associative

Syntax of Predicate Logic

4. [Lecture] The syntax of predicate logic is defined via 2 types of sorts: *terms* and *formulas*. What are terms and what are formulas? Give examples for both.



Terms talk about **objects**:

- Constants like Bob or Alice
- Variables like x, y
- Function symbols like f(x) or x+y

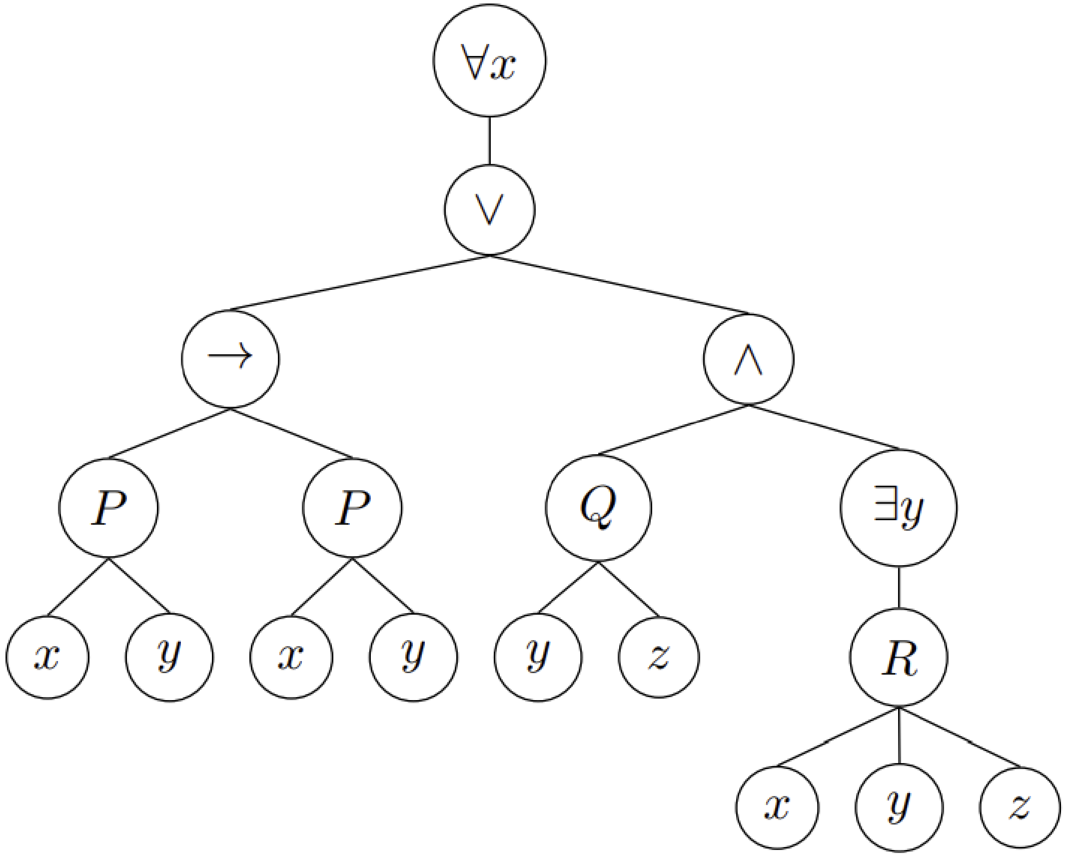
Formulas have a **truth value**:

- E.g., $\forall x(\forall y(x = y + 2))$ is a formula

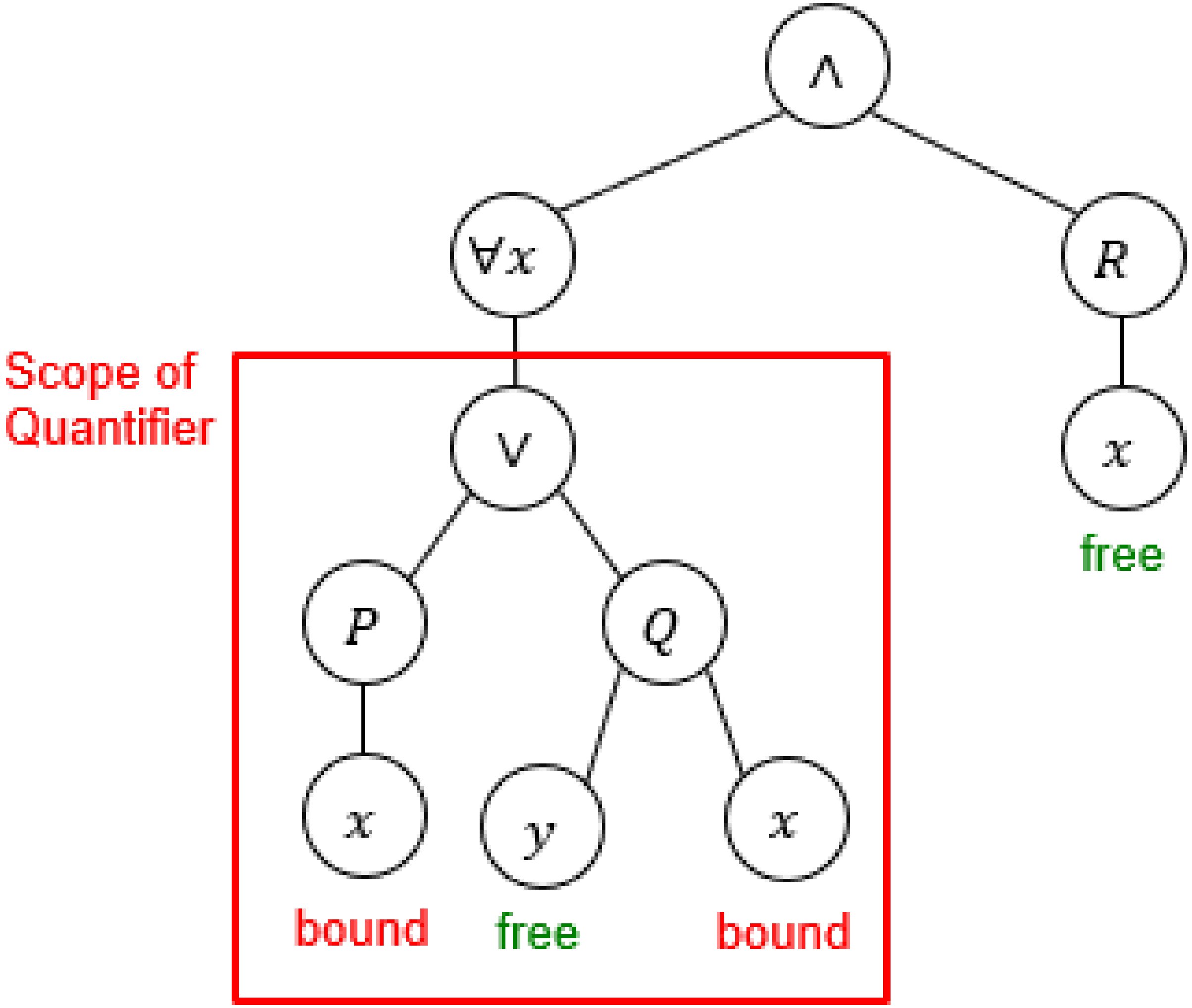
Syntax Tree

5. [Lecture] Draw a syntax tree for the following formula:

$$\forall x \left((P(x,y) \to P(x,x)) \lor (Q(y,z) \land \exists y\, R(x,y,z)) \right)$$



Free and Bound Variables

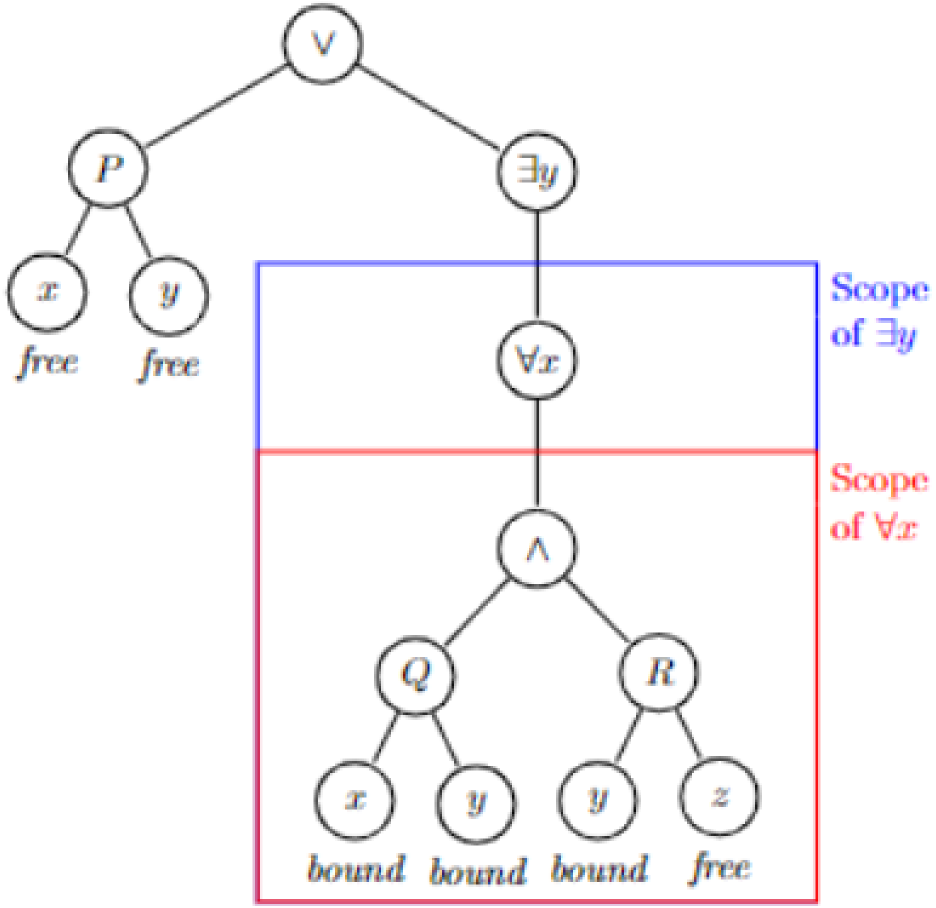


6.3 Free and Bound Variables

6. [Lecture] Given the formula

$$P(x, y) \vee \exists y \forall x \, (Q(x, y) \wedge R(y, z)),$$

construct a syntax tree for φ and determine the *scope* of its quantifiers and which occurrences of the variables are *free* and which are *bound*.



Outline

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Model M for Formulas in Predicate Logic

- Domain A
- For nullary $f \in \mathbb{F}$: concrete element $f^M \in A$
- For nullary $P \in \mathbb{P}$: true or false
- For other $f \in \mathbb{F}$: concrete function $f^M: A^n \to A$
- For $P \in \mathbb{P}$ with arity $n > 0$: subset $P^M \subseteq A^n$
- Value for free variables
  - Lookup table

Models in Predicate Logic

7. [Lecture] Give a model $\mathcal{M}$ for the following formula:

$$\varphi := \exists x \forall y P(x, y).$$

Models in Predicate Logic

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$$\mathcal{M} : \mathcal{A} = \{a, b\}$$
$$P^{\mathcal{M}} = \{(a, a), (a, b)\}$$

Semantics of Predicate Logic

- We want to know if M satisfies φ
  - $M \vDash \phi$?

- For ϕ of the form $P(t_1, t_2, \dots, t_n)$
  - Interpret all terms $t_1, \dots, t_n$ via M
  - Obtain $(a_1, a_2, \dots, a_n)$ with $a_i \in A$
  - $M \vDash P(t_1, t_2, \dots, t_n)$ iff $(a_1, a_2, \dots, a_n) \in P^M$

Semantics of Predicate Logic

- For ϕ of the form $\forall x\ \psi$
  - $M \vDash \forall x\ \psi$ iff $M \vDash_{[x \leftarrow a]} \psi$, for **all** $a \in A$

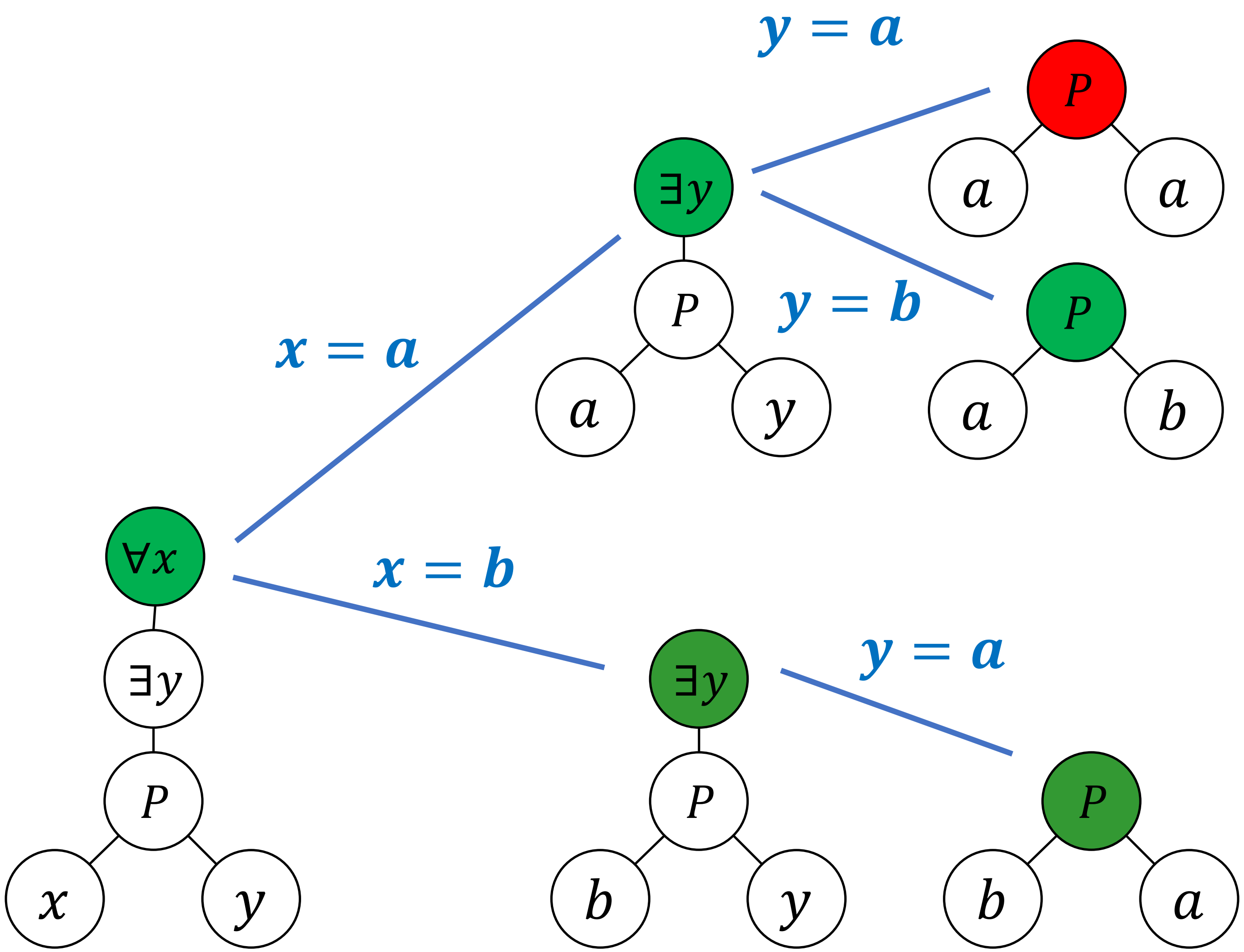
- For ϕ of the form $\exists x\ \psi$
  - $M \vDash \exists x\ \psi$ iff $M \vDash_{[x \leftarrow a]} \psi$, for **at least one** $a \in A$

- For ϕ of the form $\neg\psi$, $\psi_1 \wedge \psi_2$, $\psi_1 \vee \psi_2$, $\psi_1 \rightarrow \psi_2$
  - Like in propositional logic

$[x \leftarrow a]$ means that x is mapped to a

Evaluating a Model

- Given
  - $\phi = \forall x \exists y. P(x, y)$
  - M:
    - $A = \{a, b\}$
    - $P^M = \{(a, b), (b, a)\}$
- $M \vDash \phi$?



Evaluating a Model

8 [Lecture] Consider the formula

$$\varphi = \exists x \forall y \, (P(x,y) \rightarrow (Q(x,y) \vee R(x,y))).$$

Does the following model $\mathcal{M}$ satisfy the formula?

$\mathcal{A} = \{a, b\}$
$P^{\mathcal{M}} = \{(a,a), (a,b)\}$
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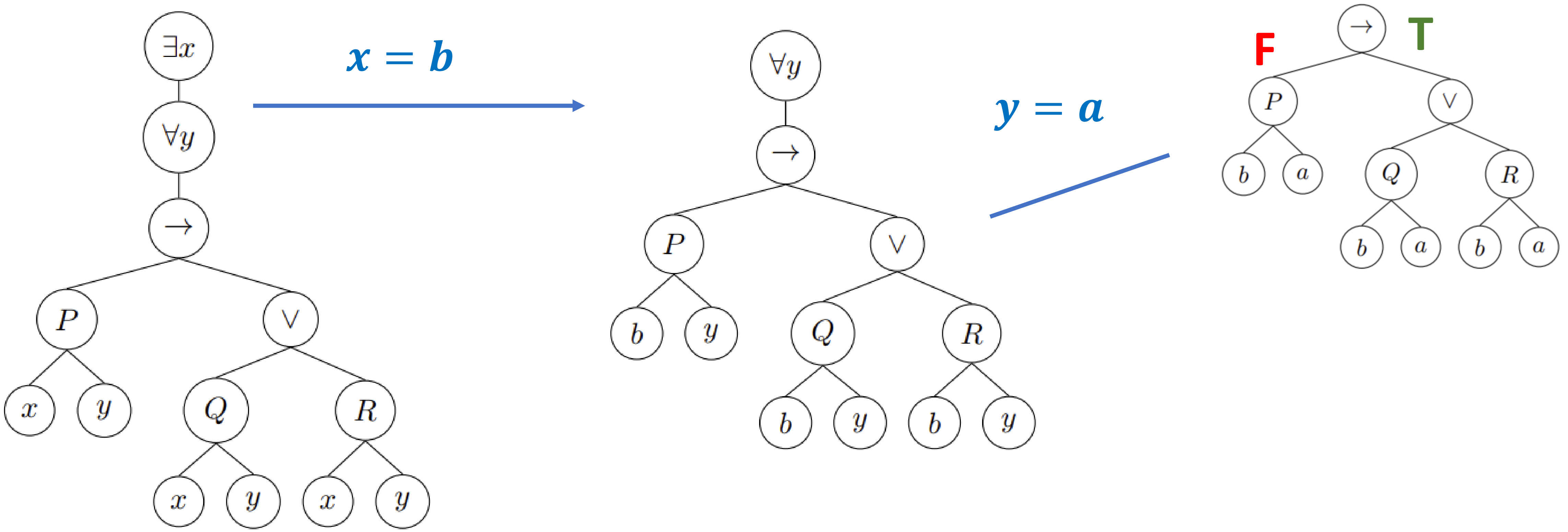

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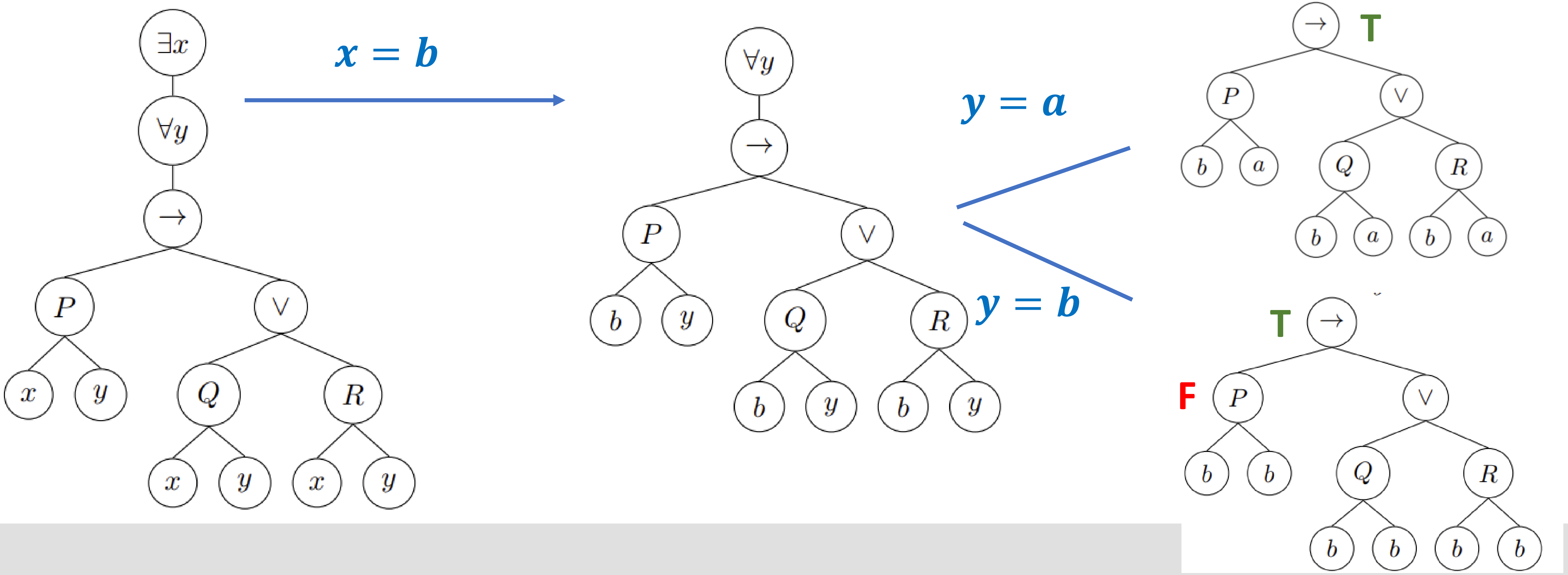

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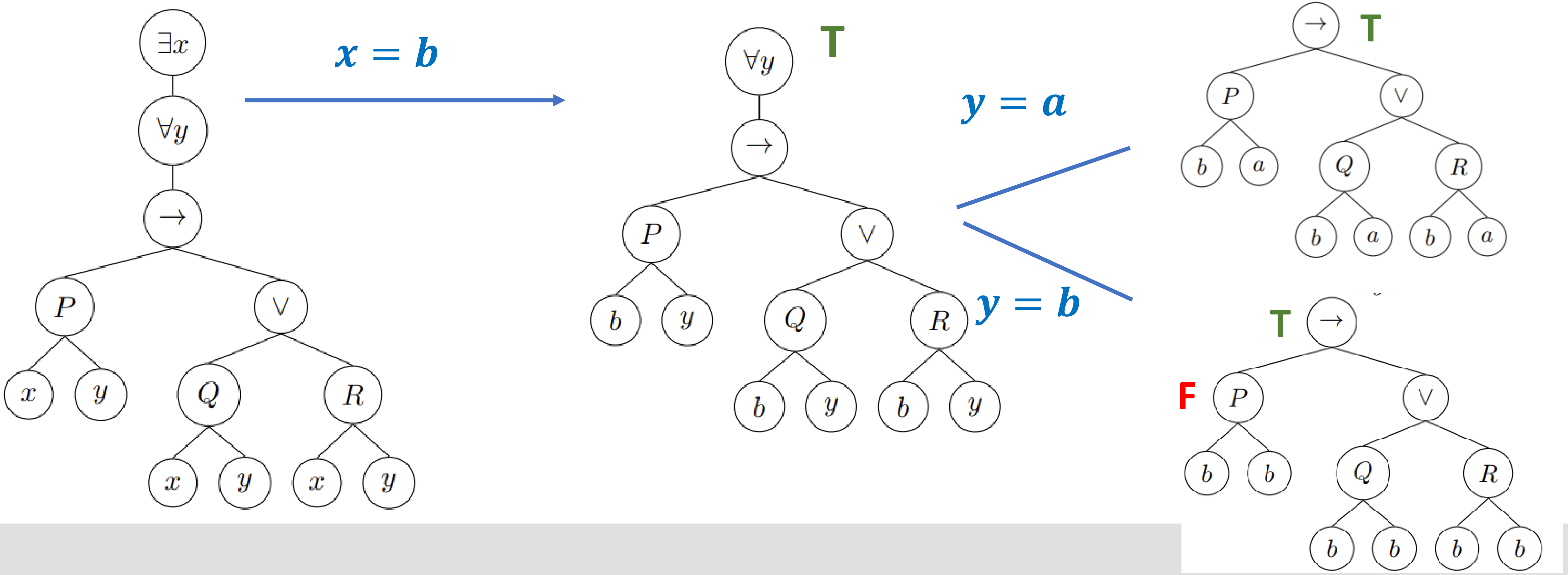

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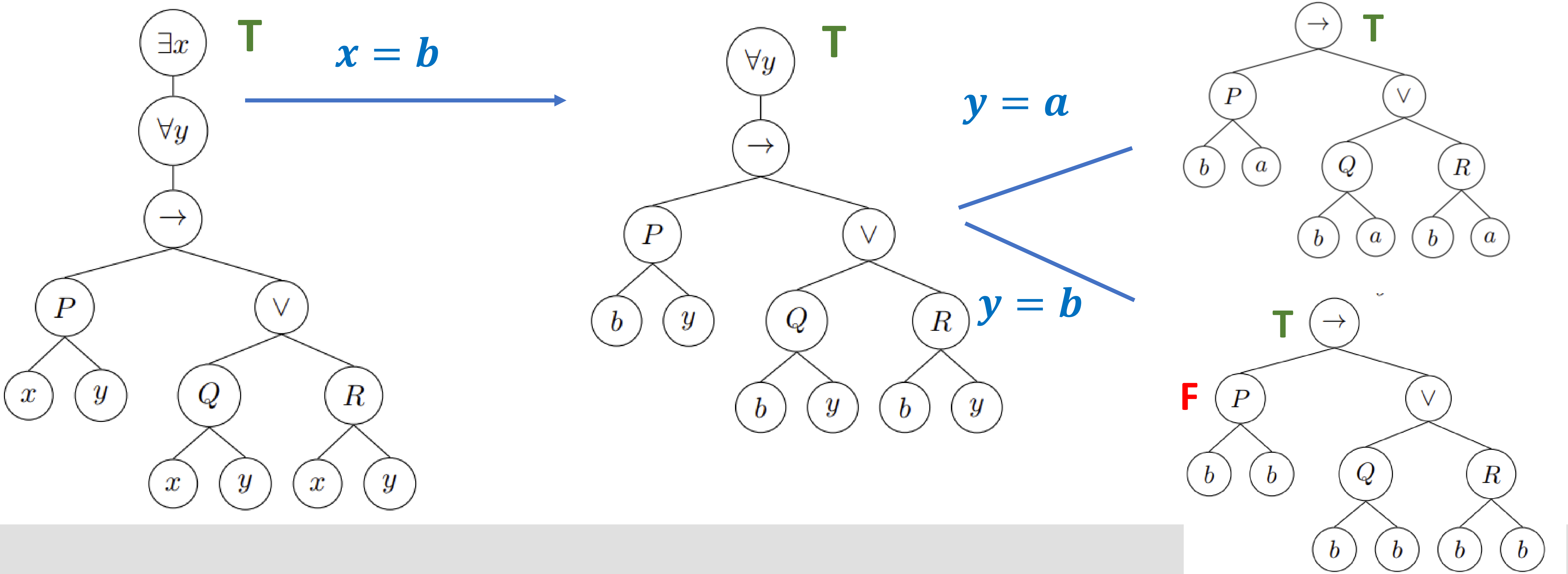

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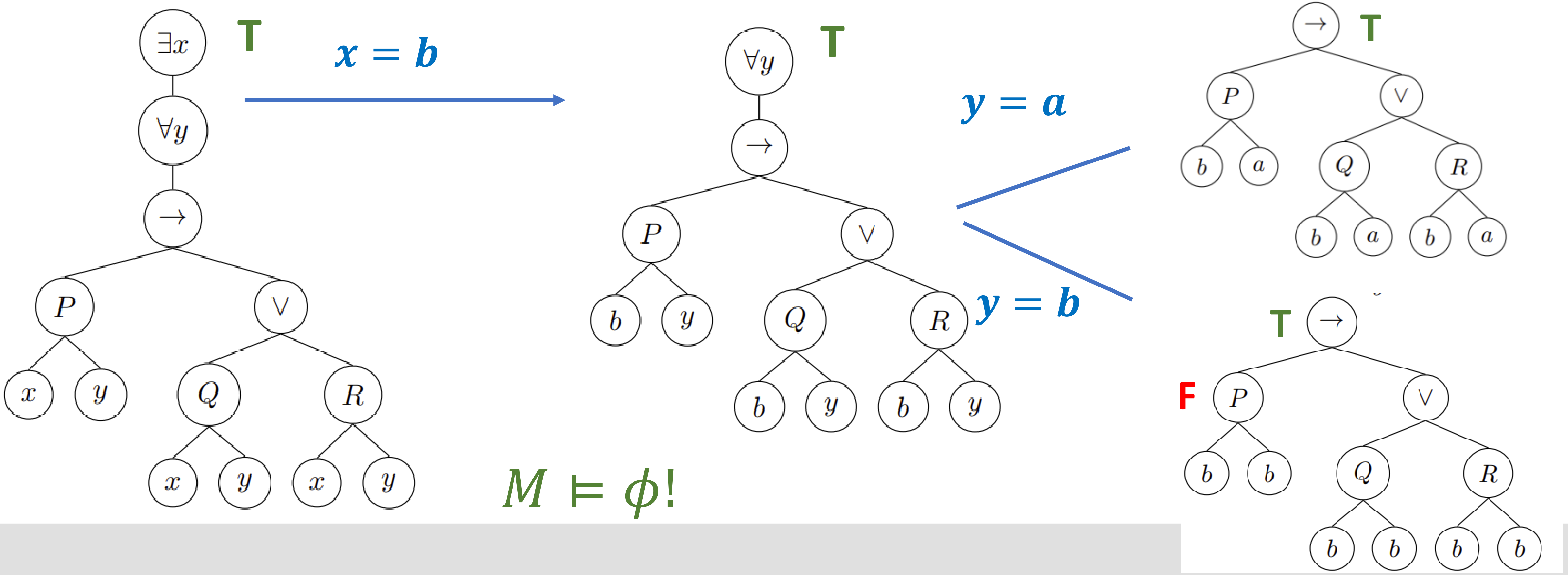
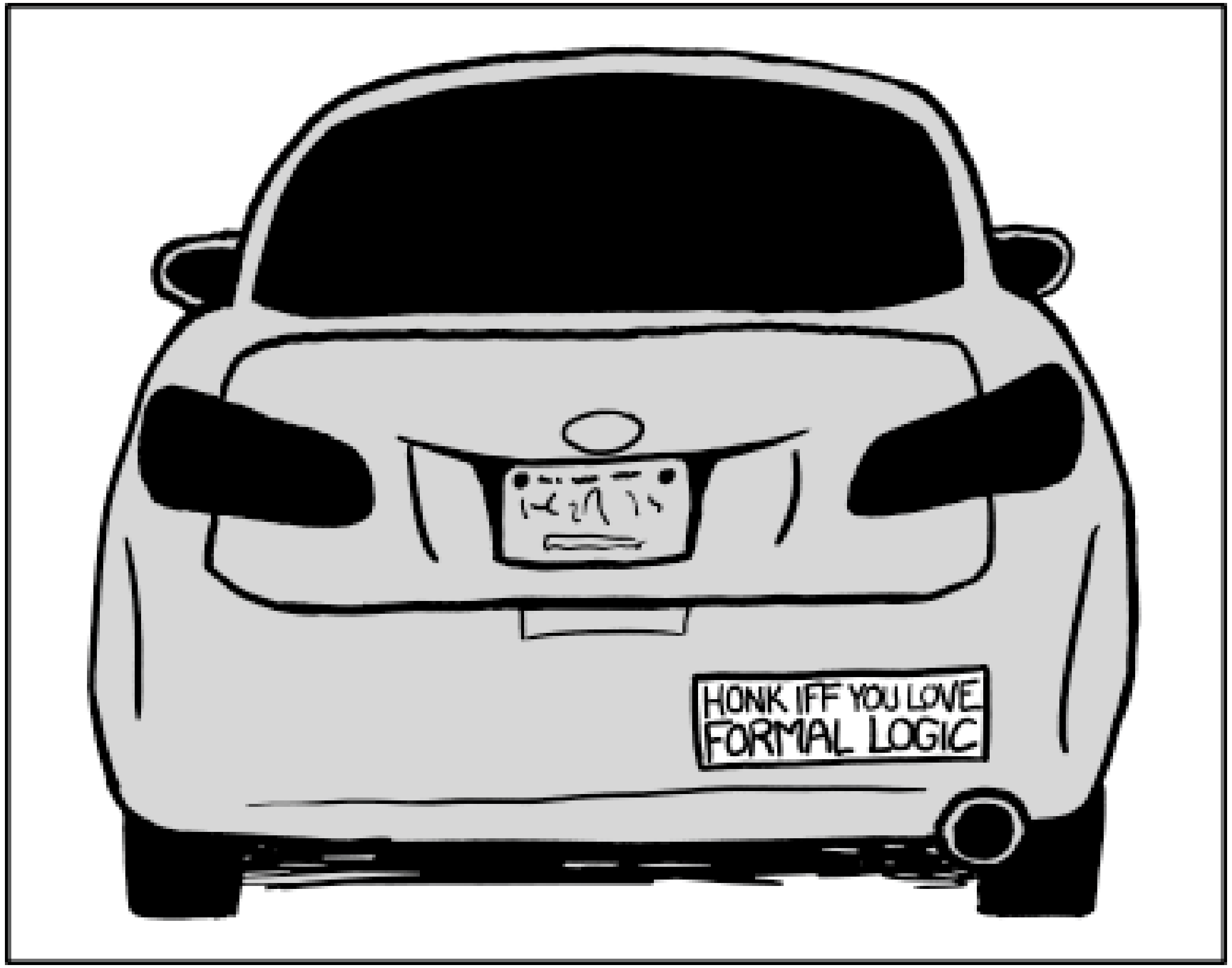


Thank You



https://xkcd.com/1033/