Example-Exam “Logic and Computability”
June 10, 2022

Name:	Matriculation number:

Rules

- Questions/Tasks are in English. Your answers may be either in German or in English, as you prefer.
- If terms are unknown to you, please ask the exam supervisor to translate them to German.
- You must work alone. Do not confer with your neighbors, or anybody else.
- You must not use any utilities, like pocket calculators, laptops, smart phones, etc.
- You must not use any documents, books, notes, scripts, etc.
- You must not use any red pen/pencil for your answers.
- Write clearly and legibly. Illegible parts will not be awarded any points.
- You must not use any of your own paper. Paper will be provided by the exam supervisor.
- Please read all tasks carefully. Also note that the exam is printed double-sided.
- If a task, or a part of a task, is not completely clear to you, please ask the exam supervisor for clarification.
- Please try to give concise, but complete answers.
- For "practical tasks" (proving something, computing something, …), write down intermediate steps to make your answer comprehensible.
- Write your name and matriculation number on the cover page. Please do not write your name and matriculation number on any other page, as the exam will be reviewed blindly.
- All your answers must be written on the printed and numbered pages. Other paper is for your notes/drafts only and will not be graded.
- You have to turn in all printed and numbered pages, even if you did not answer the questions on particular pages. You may keep the scratch paper, though.
- Concerning multiple-choice questions, be advised that for each question zero, one, several, or all answers might be correct. In general, points will only be awarded if exactly the correct answers are ticked (no more, no less).
- If you erroneously tick a multiple-choice answer and wish to "untick" it, draw a circle around the ticked box and write "No" next to it. Should you change your mind again, strike out the "No" and write "Yes" instead. Repeat, if necessary. (Try to keep corrections to a minimum, though.)
- Do not use a pencil for your final answers. Use a "permanent" pen.
- Failure to comply with any of these rules will result in appropriate consequences, e.g., a negative grade.

1. Model the following sentences as detailed as possible in propositional logic.
   (a) If a formula is unsat, it cannot be valid.
   (b) It can be proven that there exists an infinite number of primes.
   (c) A sentence is called declarative, if and only if it can be assigned a truth value.

   (a)

   $p \ldots$ A formula is unsat
   $q \ldots$ A formula is valid

   $p \to \neg q$

   (b)

   $p \ldots$ It can be proven that there exists an infinite number of primes

   p

   (c)

   $p \ldots$ A sentence is called declarative
   $q \ldots$ A sentence can be assigned a truth value

   $p \leftrightarrow q$

2. Consider the propositional formulas $\varphi = (p \vee q) \to r$, and $\psi = r \vee (\neg p \wedge \neg q)$.
   (a) Fill out the truth table for φ and ψ (and their subformulas).

p	q	r	$\neg p$	$\neg q$	$p \vee q$	$\neg p \wedge \neg q$	φ	ψ
F	**F**	**F**	**T**	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**	**F**	**T**	**T**	**T**
F	**T**	**F**	**T**	**F**	**T**	**F**	**F**	**F**
F	**T**	**T**	**T**	**F**	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**T**	**T**	**F**	**F**	**F**
T	**F**	**T**	**F**	**T**	**T**	**F**	**T**	**T**
T	**T**	**F**	**F**	**F**	**T**	**F**	**F**	**F**
T	**T**	**T**	**F**	**F**	**T**	**F**	**T**	**T**

   (b) Which of the formulas is satisfiable? Both
   (c) Which of the formulas is valid? None
   (d) Is φ equivalent to ψ? Yes
   (e) Does φ semantically entail ψ? Yes
   (f) Does ψ semantically entail φ? Yes

3. For the following sequent, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

   For proofs, clearly indicate which rule, and what assumptions and premises are used. Furthermore, clearly indicate the scope of any boxes you use. For counterexamples, give a complete model and clearly show which formulas are satisfied and which ones are not.

$$\neg q \lor p \vdash q \to (p \lor r)$$

1.	$\neg q \lor p$	prem.
2.	$\neg q$	ass.
3.	q	ass.
4.	$\bot$	$\neg_e 2, 3$
5.	p	$\bot_e 4$
6.	$p \lor r$	$\lor_i 5$
7.	$q \to (p \lor r)$	$\to_i 3-6$
8.	p	ass.
9.	q	ass.
10.	$p \lor r$	$\lor_i 5$
11.	$q \to (p \lor r)$	$\to_i 3-6$
12.	$q \to (p \lor r)$	$\lor_e 1, 2-7, 8-11$

4. For the following sequent, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

   For proofs, clearly indicate which rule, and what assumptions and premises are used. Furthermore, clearly indicate the scope of any boxes you use. For counterexamples, give a complete model and clearly show which formulas are satisfied and which ones are not.

$$\vdash (p \to q) \lor (q \to r)$$

1.	$q \lor \neg q$	LEM
2.	q	ass.
3.	p	ass.
4.	q	copy
5.	$p \to q$	$\to_i 3-4$
6.	$(p \to q) \lor (q \to r)$	$\lor_i 5$
7.	$\neg q$	ass.
8.	q	ass.
9.	$\bot$	$\neg_e 7, 8$
10.	r	$\bot_e 9$
11.	$q \to r$	$\to_i 8-10$
12.	$(p \to q) \lor (q \to r)$	$\lor_i 11$
13.	$(p \to q) \lor (q \to r)$	$\lor_e 1, 2-6, 7-12$

5. Consider the propositional formula $\varphi = (\neg(\neg a \land b) \land \neg c)$. Fill out the truth table for φ and its subformulas. Compute a CNF as well as a DNF for φ from the truth table.

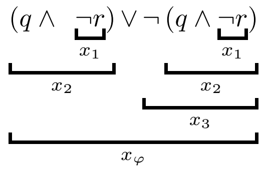
a	b	c	$\neg a$	$\neg a \land b$	$\neg(\neg a \land b)$	$\neg c$	$\varphi = (\neg(\neg a \land b) \land \neg c)$
F	**F**	**F**	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**T**	**F**	**T**	**F**	**F**
F	**T**	**F**	**T**	**T**	**F**	**T**	**F**
F	**T**	**T**	**T**	**T**	**F**	**F**	**F**
T	**F**	**F**	**F**	**F**	**T**	**T**	**T**
T	**F**	**T**	**F**	**F**	**T**	**F**	**F**
T	**T**	**F**	**F**	**F**	**T**	**T**	**T**
T	**T**	**T**	**F**	**F**	**T**	**F**	**F**

$$
\begin{aligned}
\text{CNF}: &(a \lor b \lor \neg c)\land \\
&(a \lor \neg b \lor c)\land \\
&(a \lor \neg b \lor \neg c)\land \\
&(\neg a \lor b \lor \neg c)\land \\
&(\neg a \lor \neg b \lor \neg c)
\end{aligned}
$$

$$
\begin{aligned}
\text{DNF}: &(\neg a \land \neg b \land \neg c)\lor \\
&(a \land \neg b \land \neg c)\lor \\
&(a \land b \land \neg c)
\end{aligned}
$$

6. Apply Tseitin's encoding to the following formula: $\varphi = (q \land \neg r) \lor \neg(q \land \neg r)$. For each variable you introduce, clearly indicate which subformula of φ it represents. Use the following equivalences:

$$
\begin{aligned}
\chi \leftrightarrow (\varphi \lor \psi) &\quad\Leftrightarrow\quad (\neg\varphi \lor \chi) \land (\neg\psi \lor \chi) \land (\neg\chi \lor \varphi \lor \psi) \\
\chi \leftrightarrow (\varphi \land \psi) &\quad\Leftrightarrow\quad (\neg\chi \lor \varphi) \land (\neg\chi \lor \psi) \land (\neg\varphi \lor \neg\psi \lor \chi) \\
\chi \leftrightarrow \neg\varphi &\quad\Leftrightarrow\quad (\neg\chi \lor \neg\varphi) \land (\varphi \lor \chi)
\end{aligned}
$$



$$
\begin{aligned}
CNF(\varphi) = x_\varphi \land & \\
&(\neg x_2 \lor x_\varphi) \land (\neg x_3 \lor x_\varphi) \land (\neg x_\varphi \lor x_2 \lor x_3)\land \\
&(\neg x_3 \lor \neg x_2) \land (x_3 \lor x_2)\land \\
&(\neg x_2 \lor q) \land (\neg x_2 \lor x_1) \land (\neg q \lor \neg x_1 \lor x_2)\land \\
&(\neg x_1 \lor \neg r) \land (x_1 \lor r)
\end{aligned}
$$

7. Consider the domain $A = \{Spain, France, Italy, Germany\}$ and the two different symbolic encodings for A given below. Which one gives a shorter symbolic representation for the set $B = \{France, Italy\}$? Illustrate your answer by giving the representing formulas for B in both encodings.

Encoding 1

Element	v_1	v_0
Spain	0	0
France	1	0
Italy	0	1
Germany	1	1

Encoding 2

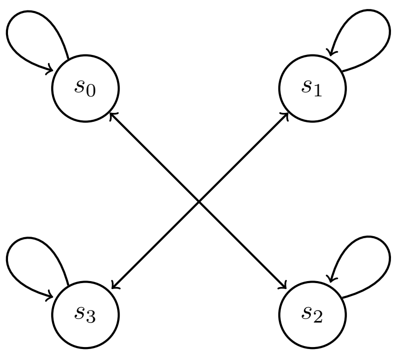
Element	v_1	v_0
Spain	0	0
France	1	0
Italy	1	1
Germany	0	1

We give a symbolic representation for B using v_1 and v_0 and compare the lengths of the formulas.

Encoding 1: $b = (v_1 \wedge \neg v_0) \vee (\neg v_1 \wedge v_0)$ Encoding 2: $b = v_1$

Comparing these two encodings shows that '**Encoding 2**' yields a shorter representation for B.

8. Find a *symbolic encoding* for the set of initial states and the *transition relation* of the following *transition system* and simplify your formulas. Use a binary encoding to encode the states, e.g., encode the state s_2 with the formula $v_1 \wedge \neg v_0$.

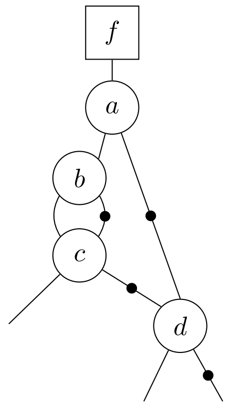


$$S_0 = \emptyset$$

$$\begin{aligned} R = \neg v_1 \wedge \neg v_0 \wedge\ & (\neg v_1' \wedge \neg v_0' \vee v_1' \wedge \neg v_0') \vee \\ \neg v_1 \wedge v_0 \wedge\ & (\neg v_1' \wedge v_0' \vee v_1' \wedge v_0') \vee \\ v_1 \wedge \neg v_0 \wedge\ & (v_1' \wedge \neg v_0' \vee \neg v_1' \wedge \neg v_0') \vee \\ v_1 \wedge v_0 \wedge\ & (v_1' \wedge v_0' \vee \neg v_1' \wedge v_0') \end{aligned}$$

9. Given the *Binary Decision Diagram (BDD)* below. State the formula f that is represented by the BDD.

   Note: Else-edges are marked with circles. Filled circles represent the *complemented* attribute. Dangling edges are assumed to point to the constant node `true`.



   We state the formula as a DNF:

$$f = (a \land b \land c) \lor (a \land \neg b \land \neg c \land d) \lor (\neg a \land \neg d)$$

10. Consider the following declarative sentence:

    "For every natural number it holds that it is prime if and only if there is no smaller natural number, except for 1, that divides it."

    Model this sentence with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

    $\mathcal{A} = \mathbb{N}$

    $P(n)$ ... n is prime

    $D(m, n)$ ... m divides n

$$\forall n \; ((P(n) \leftrightarrow \neg \exists m (m > 1) \land D(m, n)))$$

11. For the following sequent, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

    For proofs, clearly indicate which rule, and what assumptions and premises are used. Furthermore, clearly indicate the scope of any boxes you use. For counterexamples, give a complete model and clearly show which formulas are satisfied and which ones are not.

$$\forall x(P(x) \lor Q(x)), \quad \forall x(P(x) \to R(z)), \quad \forall y(Q(y) \to R(z)) \quad \vdash \quad R(z)$$

1.	$\forall x\ (P(x) \lor Q(x))$	prem.
2.	$\forall x\ (P(x) \to R(z))$	prem.
3.	$\forall x\ (Q(x) \to R(z))$	prem.
4.	$P(x_0) \lor Q(x_0)$	$\forall_e 1$
5.	$P(x_0) \to R(z)$	$\forall_e 2$
6.	$Q(x_0) \to R(z)$	$\forall_e 3$
7.	$P(x_0)$	ass.
8.	$R(z)$	$\to_e 5, 7$
9.	$Q(x_0)$	ass.
10.	$R(z)$	$\to_e 6, 4$
11.	$R(z)$	$\lor_e 4, 7-8, 9-10$

12. For the following sequent, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

    For proofs, clearly indicate which rule, and what assumptions and premises are used. Furthermore, clearly indicate the scope of any boxes you use. For counterexamples, give a complete model and clearly show which formulas are satisfied and which ones are not.

$$\forall x\ (Q(x) \to R(x)), \quad \exists x\ (P(x) \land Q(x)) \quad \vdash \quad \exists x\ (P(x) \land R(x))$$

1.	$\forall x\ (Q(x) \to R(x))$	prem.
2.	$\exists x\ (P(x) \to Q(z))$	prem.
3.	x_0 $P(x_0) \land Q(x_0)$	ass.
4.	$P(x_0)$	$\land_e 3$
5.	$Q(x_0)$	$\land_e 3$
6.	$Q(x_0) \to R(x_0)$	$\forall_e 1$
7.	$R(x_0)$	$\to_e 5, 6$
8.	$P(x_0) \land R(x_0)$	$\land_i 4, 7$
9.	$\exists x(P(x) \land R(x))$	$\exists_i 8$
10.	$\exists x(P(x) \land R(x))$	$\exists_e 2, 3-9$

13. Consider the following formula in $\mathcal{T}_{EUF}$.

$$\varphi_{EUF} \quad := \quad f(g(x),h(y)) = a \ \lor \ b = f(u,v) \ \rightarrow \ k(a,b) = u \land v = k(x,y)$$

Use Ackermann’s reduction to compute an equisatisfiable formula in $\mathcal{T}_E$.

$f \ \ldots \ f_{gh}, f_{uv}$
$g \ \ldots \ g_x$
$h \ \ldots \ h_y$
$k \ \ldots \ k_{ab}, k_{xy}$

$$\begin{aligned}\varphi_{FC} &= ((g = u \land h = v) \rightarrow f_{gh} = f_{uv} \land \\ &\quad ((a = x \land b = y) \rightarrow k_{ab} = k_{xy} \\ \hat{\varphi} &= f_{gh} = a \lor b = f_{uv} \rightarrow k_{ab} = u \land v = k_{xy} \\ \varphi_E &= \varphi_{FC} \land \hat{\varphi}\end{aligned}$$

14. In the following text fill the blanks with the missing word(s).

The Ackermann’s reduction is used to reduce a formula φ_{in} in the theory of equality and uninterpreted functions to a formula in the theory of equality that is equisatisfiable. Two formulas are equisatisfiable if either both of them are satisfiable or both are not satisfiable. The algorithm adds explicit constraints to the formula φ_{in} to enforce functional consistency. These constraints say, that $\forall \bar{x} \forall \bar{y} \ (\bigwedge_i x_i = y_i) \rightarrow f(\bar{x}) = f(\bar{y})$. The resulting equisatisfiable formula consists of two parts and is of the form: $\varphi_{out} := \varphi_C \land \hat{\varphi_{in}}$. The right part of the formula $\hat{\varphi_{in}}$ describes the flattening original formula in which we replace function applications with fresh variables.

15. Use the DPLL algorithm with conflict-driven clause learning to determine whether or not the set of clauses given is satisfiable. Decide variables in alphabetical order starting with the *negative* phase. For conflicts, draw conflict graphs after the end of the table, and add the learned clause to the table.
If the set of clauses resulted in `SAT`, give a satisfying model. If the set of clauses resulted in `UNSAT`, give a resolution proof that shows that the conjunction of the clauses from the table is unsatisfiable.

Clause 1: $\{b, d\}$

Clause 2: $\{b, c\}$

Clause 3: $\{\neg b, \neg e\}$

Clause 4: $\{\neg a, \neg c\}$

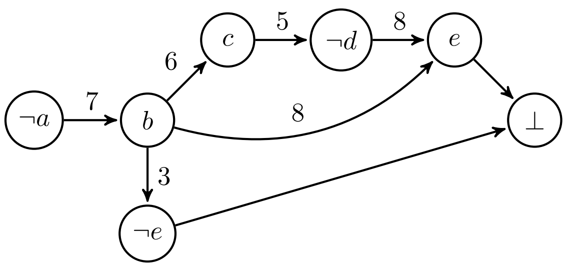
Clause 5: $\{\neg c, \neg d\}$

Clause 6: $\{\neg b, c\}$

Clause 7: $\{a, b\}$

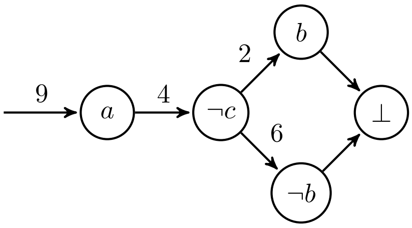
Clause 8: $\{\neg b, d, e\}$

Step	1	2	3	4	5	6
Decision Level	0	1	1	1	1	1
Assignment	-	¬a	¬a,b	¬a,b,c	¬a,b,c,¬d	¬a,b,c,¬d, ¬e
Cl. 1: b, d	b, d	b, d	✓	✓	✓	✓
Cl. 2: b, c	b, c	b, c	✓	✓	✓	✓
Cl. 3: $\neg b, \neg e$	$\neg b, \neg e$	$\neg b, \neg e$	$\neg e$	$\neg e$	$\neg e$	$\neg e$
Cl. 4: $\neg a, \neg c$	$\neg a, \neg c$	✓	✓	✓	✓	✓
Cl. 5: $\neg c, \neg d$	$\neg c, \neg d$	$\neg c, \neg d$	$\neg c, \neg d$	$\neg d$	✓	✓
Cl. 6: $\neg b, c$	$\neg b, c$	$\neg b, c$	c	✓	✓	✓
Cl. 7: a, b	a, b	b	✓	✓	✓	✓
Cl. 8: $\neg b, d, e$	$\neg b, d, e$	$\neg b, d, e$	d, e	d, e	e	{} ✗
BCP	-	b	c	$\neg d$	$\neg e$	-
PL	-	-	-	-	-	-
Decision	$\neg a$	-	-	-	-	-



$$\dfrac{\dfrac{\dfrac{\dfrac{8.\ \neg b \lor d \lor e \qquad 3.\ \neg b \lor \neg e}{\neg b \lor d} \qquad 5.\ \neg c \lor \neg d}{\neg b \lor \neg c} \qquad 6.\ \neg b \lor c}{\neg b} \qquad 7.\ a \lor b}{a}$$

Step	7(0)	8	9	10
Decision Level	0	0	0	0
Assignment	-	a	$a, \neg c$	$a, \neg c, \neg b$
Cl. 1: b, d	b, d	b, d	b, d	d
Cl. 2: b, c	b, c	b, c	b	{} ✗
Cl. 3: $\neg b, \neg e$	$\neg b, \neg e$	$\neg b, \neg e$	$\neg b, \neg e$	✓
Cl. 4: $\neg a, \neg c$	$\neg a, \neg c$	$\neg c$	✓	✓
Cl. 5: $\neg c, \neg d$	$\neg c, \neg d$	$\neg c, \neg d$	✓	✓
Cl. 6: $\neg b, c$	$\neg b, c$	$\neg b, c$	$\neg b$	✓
Cl. 7: a, b	a, b	✓	✓	✓
Cl. 8: $\neg b, d, e$	$\neg b, d, e$	$\neg b, d, e$	$\neg b, d, e$	✓
Cl. LC: 9: a	a	✓	✓	✓
BCP	a	$\neg c$	$\neg b$	-
PL	-	-	-	-
Decision	-	-	-	UNSAT



$$\dfrac{\dfrac{\dfrac{2.\ b \lor c \qquad 6.\ \neg b \lor c}{c} \qquad 4.\ \neg a \lor \neg c}{\neg a} \qquad 9.\ a}{\bot}$$