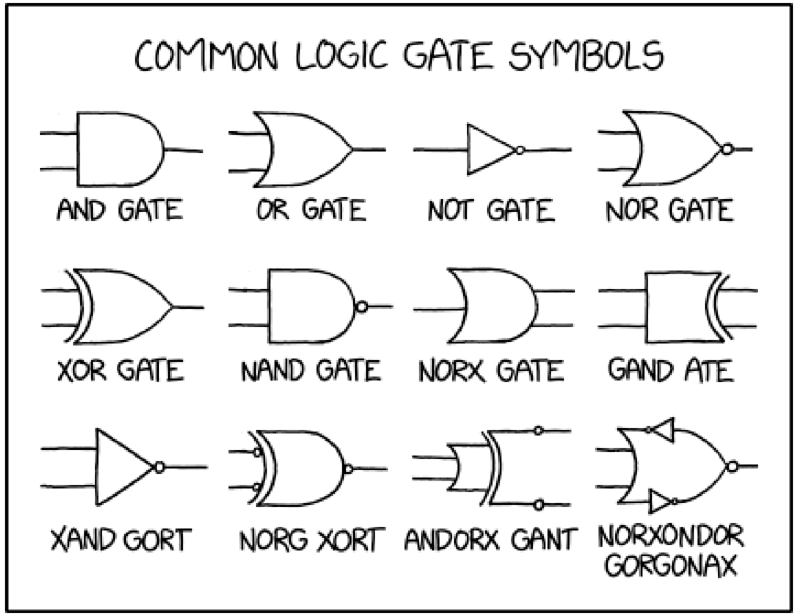
Logic and Computability

Lecture 1

Propositional Logic

Bettina Könighofer
bettina.koenighofer@lamarr.at

Stefan Pranger
stefan.pranger@iaik.tugraz.at



https://xkcd.com/2497/

Why are we interested in Logic?

- Our aim is to formulate specifications and to model systems such that we can *reason* about them *formally*.

Why are we interested in Logic?

- Our aim is to formulate specifications and to model systems such that we can *reason* about them *formally*.

- Use logic to write **unambiguous specifications**

- Formal verification
  - Technique to prove that system satisfies the specification

- Formal synthesis
  - Technique to automatically synthesize system from its formal specification

Why are we interested in Logic?

- Prove that the argumentation is valid:
  - Show that the sentence after the "*therefore*" **logically follows** from the sentences before.

Why are we interested in Logic?

- Prove that the argumentation is valid:
  - Show that the sentence after the "*therefore*" **logically follows** from the sentences before.

1. If the plane arrives late and there are no taxis at the airport, then Alice is late for her appointment.
2. Alice is not late for her appointment.
3. The plane did arrive late.
4. *Therefore*, there were taxis at the airport.

Why are we interested in Logic?

- Prove that the argumentation is valid:
  - Show that the sentence after the "*therefore*" **logically follows** from the sentences before.

1. If the plane arrives late and there are no taxis at the airport, then Alice is late for her appointment.
2. Alice is not late for her appointment.
3. The plane did arrive late.
4. *Therefore*, there were taxis at the airport.

Logical Structure

If p and not q, then r. Not r. p. *Therefore*, q

Why are we interested in Logic?

- Prove that the argumentation is valid:
  - Show that the sentence after the "*therefore*" **logically follows** from the sentences before.

1. If the sun is shining and John does not have his sun creme with him, then he will get sunburn.
2. John is not sunburn.
3. The sun is shining.
4. *Therefore*, John has his sunscreen with him.

Why are we interested in Logic?

- Prove that the argumentation is valid:
  - Show that the sentence after the "*therefore*" **logically follows** from the sentences before.

1. If the sun is shining and John does not have his sun creme with him, then he will get sunburn..
2. John is not sunburn.
3. The sun is shining.
4. *Therefore*, John has his sunscreen with him.

Logical Structure (same as before)

If p and not q, then r. Not r. p. *Therefore*, q

Why are we interested in Logic?

- Prove that the argumentation is valid:
  - Show that the sentence after the “*therefore*” **logically follows** from the sentences before.

1. If the sun is shining and John does not have his sun creme with him, then he will get sunburn..
2. John is not sunburn.
3. The sun is shining.
4. *Therefore*, John has his sunscreen with him.

Logical Structure (same as before)

If p and not q, then r. Not r. p. *Therefore*, q

Next Week:
How can we prove that?

Outline

- Declarative Sentences
- Syntax
  - Symbols & Grammar
  - Parse Tree
- Semantics
  - Meaning
  - Models
  - Truth Tables
  - Validity, Satisfiability, Entailment & Equivalence
- Examples

Declarative Sentence

- Statements that are *true* or *false*
  - Sentences state a fact

Declarative Sentence

- Statements that are *true* or *false*
  - Sentences state a fact
- Simple
  - "The sun is shining."
  - "Tomorrow is Wednesday."
  - "Santa Clause lives on the North Pole."
  - → Propositional Atoms

Declarative Sentence

- Statements that are *true* or *false*
  - Sentences state a fact
- Simple
  - "The sun is shining."
  - "Tomorrow is Wednesday."
  - "Santa Clause lives on the North Pole."
  - → Propositional Atoms
- With Structure
  - "I like pizza and not ice creme."
    - p: "I like pizza.", q: "I like ice creme.", $\mathbf{p} \land \neg\mathbf{q}$

Non-Declarative Sentence

- Questions
  - “What time is it?”
- Commands
  - “Do your homework!”
- Exclamations
  - “Oh my god!”
- Various others
  - “Ready, steady, go.”
  - “Good night, my friend.”
  - “May the force be with you.”
  - “Live long and prosper”
  - “Give peace a chance!”

Outline

- Declarative Sentences ✓

- **Syntax**
  - Symbols & Grammar
  - Parse Tree

- Semantics
  - Meaning
  - Models
  - Truth Tables
  - Validity, Satisfiability, Entailment & Equivalence

- Examples

Syntax of Propositional Logic

- Elements/Symbols
  - Atomic propositions/ propositional variables **p**, **q**, **r** ...
  - Conjunction **∧**, Disjunction **∨**, Negation **¬**, Implication →, Equivalence ≡ (or ↔) , Parentheses **()**

Syntax of Propositional Logic

- Elements/Symbols
  - Atomic propositions/ propositional variables $\mathbf{p}, \mathbf{q}, \mathbf{r}$ ...
  - Conjunction **∧**, Disjunction **∨**, Negation **¬**, Implication →, Equivalence ≡ (or ↔) , Parentheses **()**
- Syntax defines the set of strings that defines well-formed formulas
  - E.g., "$\boldsymbol{pq}()(\wedge)$" is not a well formed formula

Syntax of Propositional Logic

- Elements/Symbols
  - Atomic propositions/ propositional variables **p**, **q**, **r** ...
  - Conjunction **∧**, Disjunction **∨**, Negation **¬**, Implication →, Equivalence ≡ (or ↔) , Parentheses **()**
- Syntax defines the set of strings that defines well-formed formulas
  - E.g., "$\boldsymbol{pq}()(\wedge)$" is not a well formed formula
- Grammar

$$\varphi := < \text{automic proposition} > \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi \mid \varphi \rightarrow \varphi \mid \varphi \leftrightarrow \varphi \mid (\varphi)$$

Parentheses and Operator Precedence

- Without parentheses:

Highest	$\neg$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	Lowest

  - Right-associative:

    $p \rightarrow q \rightarrow r$ means $p \rightarrow (q \rightarrow r)$

Parentheses and Operator Precedence

- Without parentheses:

Highest $\neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow$ Lowest

  - Right-associative:

    p → q → r means p → (q → r)

Example:

$p \wedge q \wedge r \rightarrow p \rightarrow r$
$\Rightarrow \big((p \wedge q) \wedge r\big) \rightarrow (p \rightarrow r)$

$\neg p \wedge q$
$\Rightarrow (\neg p) \wedge q$

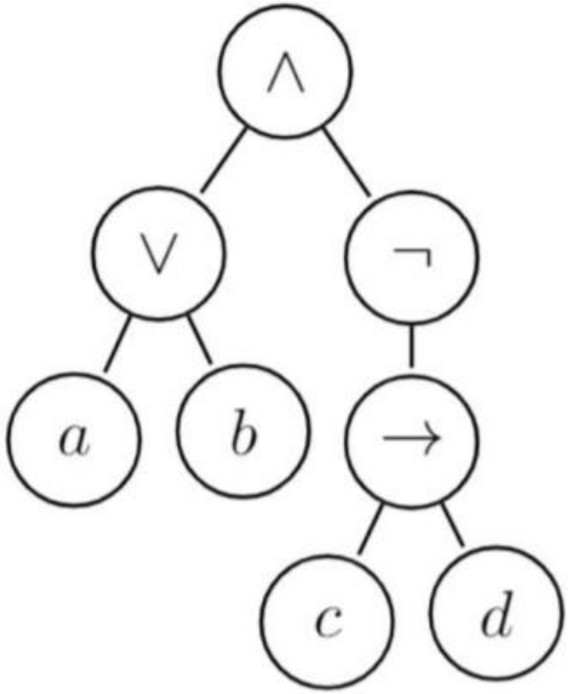
Parse Tree

- Used to show that a string is *well-formed* formula

- Formula is *well-formed*, if
  - all leaves are labelled with atomic propositions and
  - all other nodes are labelled with logical operators

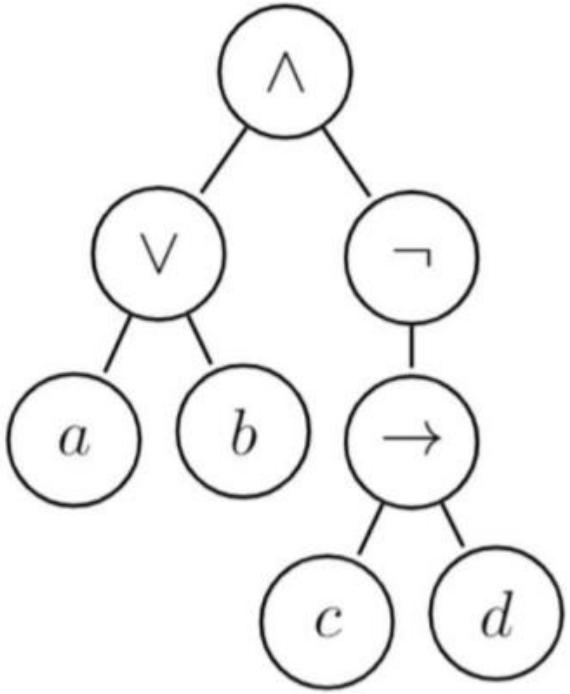
Parse Tree - Example

- Used to show that a string is *well-formed* formula
- Example: $\varphi = (a \lor b) \land (\neg(c \rightarrow d))$



Parse Tree - Example

- Used to show that a string is *well-formed* formula
- Example: $\varphi = (a \vee b) \wedge (\neg(c \to d))$



φ is a *well-formed* formula

Outline

- Declarative Sentences ✔
- Syntax ✔
  - Symbols & Grammar
  - Parse Tree
- **Semantics**
  - Meaning
  - Models
  - Truth Tables
  - Validity, Satisfiability, Entailment & Equivalence
- Examples

Semantics of Propositional Logic

- Truth is a *semantic notion*, assigns meaning to formulas.

Semantics of Propositional Logic

- Truth is a *semantic notion*, assigns meaning to formulas.

- A formula is true or false based on the truth/falsehood of the statements it comprises and the meaning of the logical operators

Semantics of Propositional Logic

- Truth is a *semantic notion*, assigns meaning to formulas.

- A formula is true or false based on the truth/falsehood of the statements it comprises and the meaning of the logical operators

- The semantic defines the rules for the logical operators via truth tables

φ	ψ	$\varphi \wedge \psi$
F	F	F
F	T	F
T	F	F
T	T	T

φ	ψ	$\varphi \vee \psi$
F	F	F
F	T	T
T	F	T
T	T	T

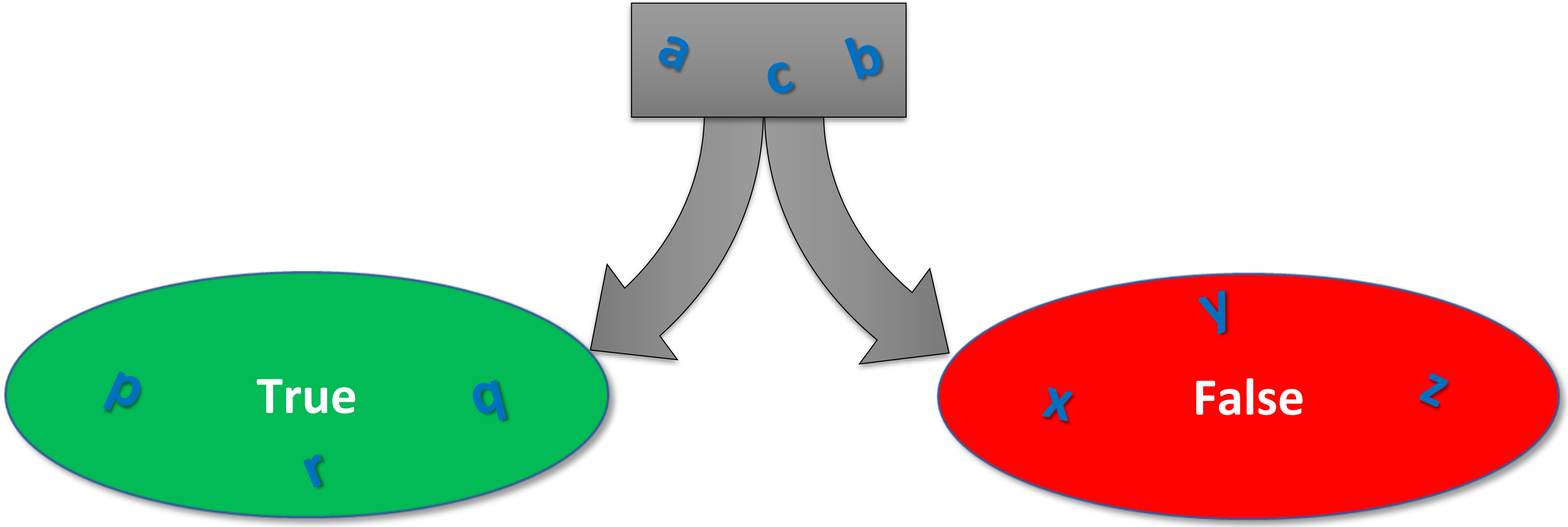
φ	$\neg\varphi$
F	T
T	F

φ	ψ	$\varphi \rightarrow \psi$
F	F	T
F	T	T
T	F	F
T	T	T

φ	ψ	$\varphi \leftrightarrow \psi$
F	F	T
F	T	F
T	F	F
T	T	T

Models $\mathcal{M}$

- Model ≅ Valuation ≅ Interpretation ≅ Assignment
- Assignment: $\{Atomic\ propositions\} \mapsto \{\mathbf{True}, \mathbf{False}\}$



Models $\mathcal{M}$

- Model ≅ Valuation ≅ Interpretation ≅ Assignment
- Assignment: $\{Atomic\ propositions\} \mapsto \{\mathbf{True}, \mathbf{False}\}$

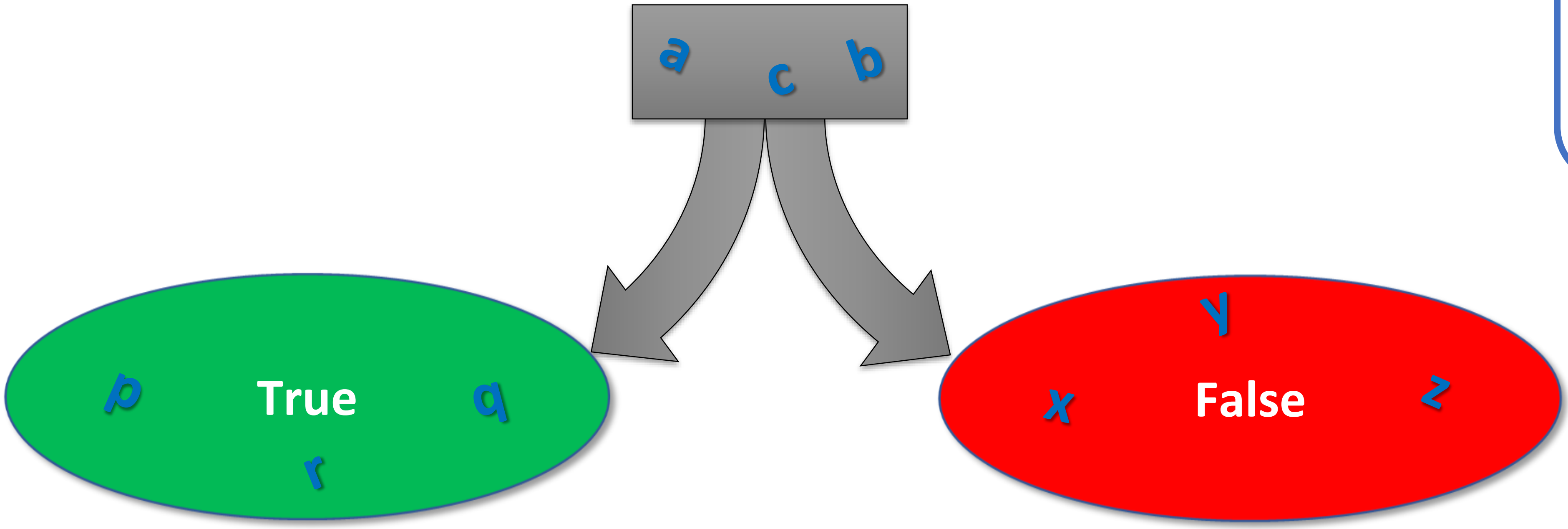
Notation

True:

T,1,⊤ (LaTeX: \top)

False:

F,0, ⊥ (LaTeX: \bot)



Models $\mathcal{M}$

- $\varphi^{\mathcal{M}}$: φ is evaluated under $\mathcal{M}$
- Satisfying Model: $\mathcal{M} \vDash \varphi$
  - Model satisfies the formula

$\vDash$ … Semantic Entailment Relation

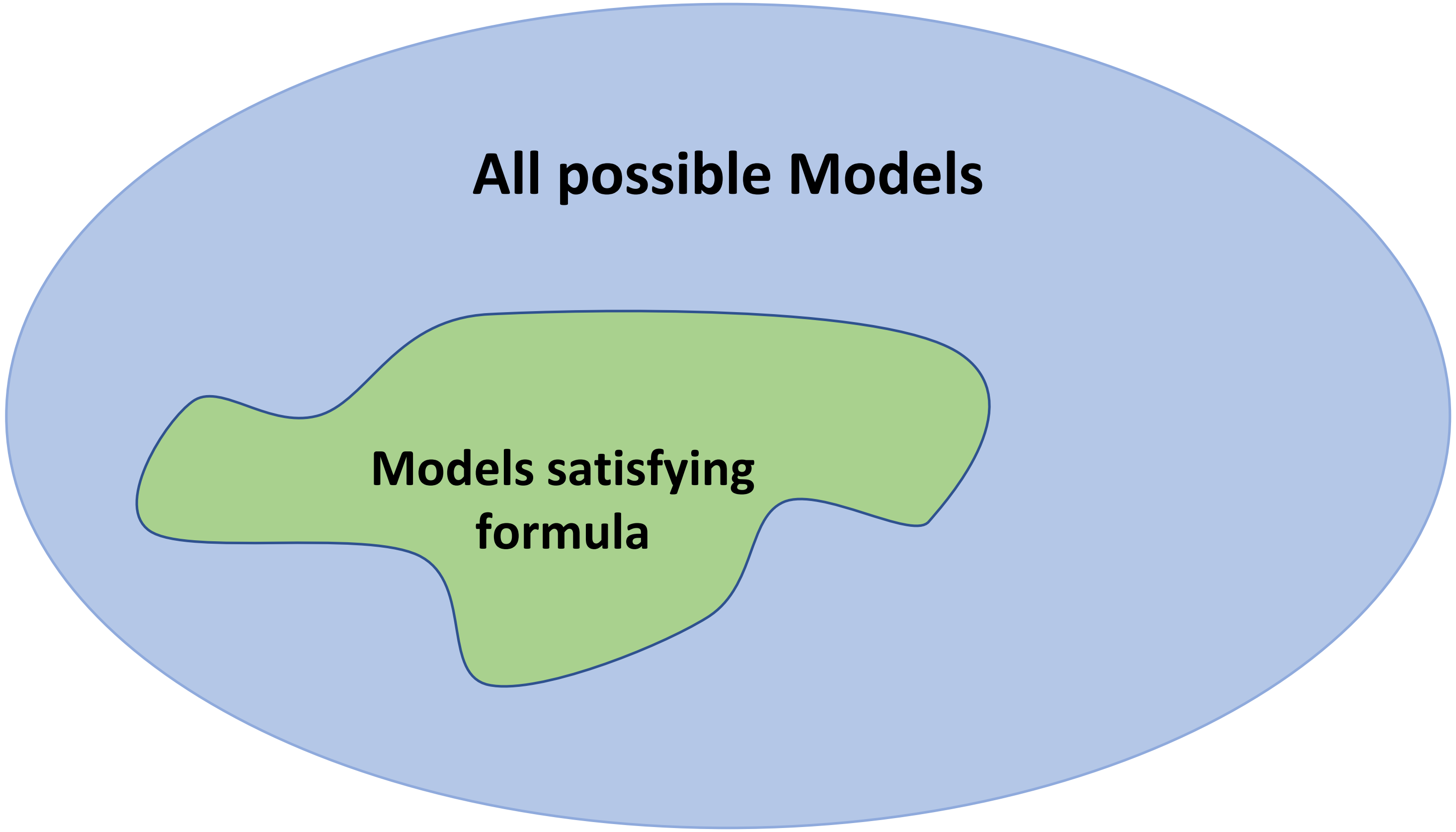
- Falsifying Model: $\mathcal{M} \nvDash \varphi$
  - Model does not satisfy the formula

Notions of Semantics in prop. Logic

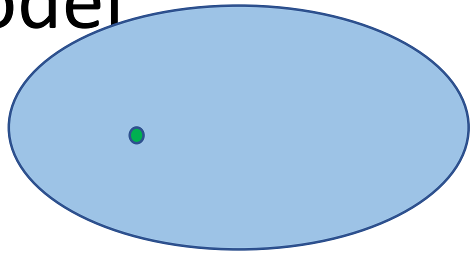
- Satisfiability
- Validity
- Semantic Entailment
- Equivalence

Satisfiability

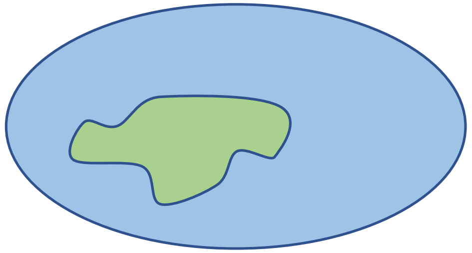
- At least one model satisfies formula



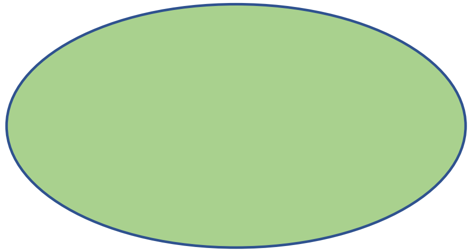
- One Model



- Several Models

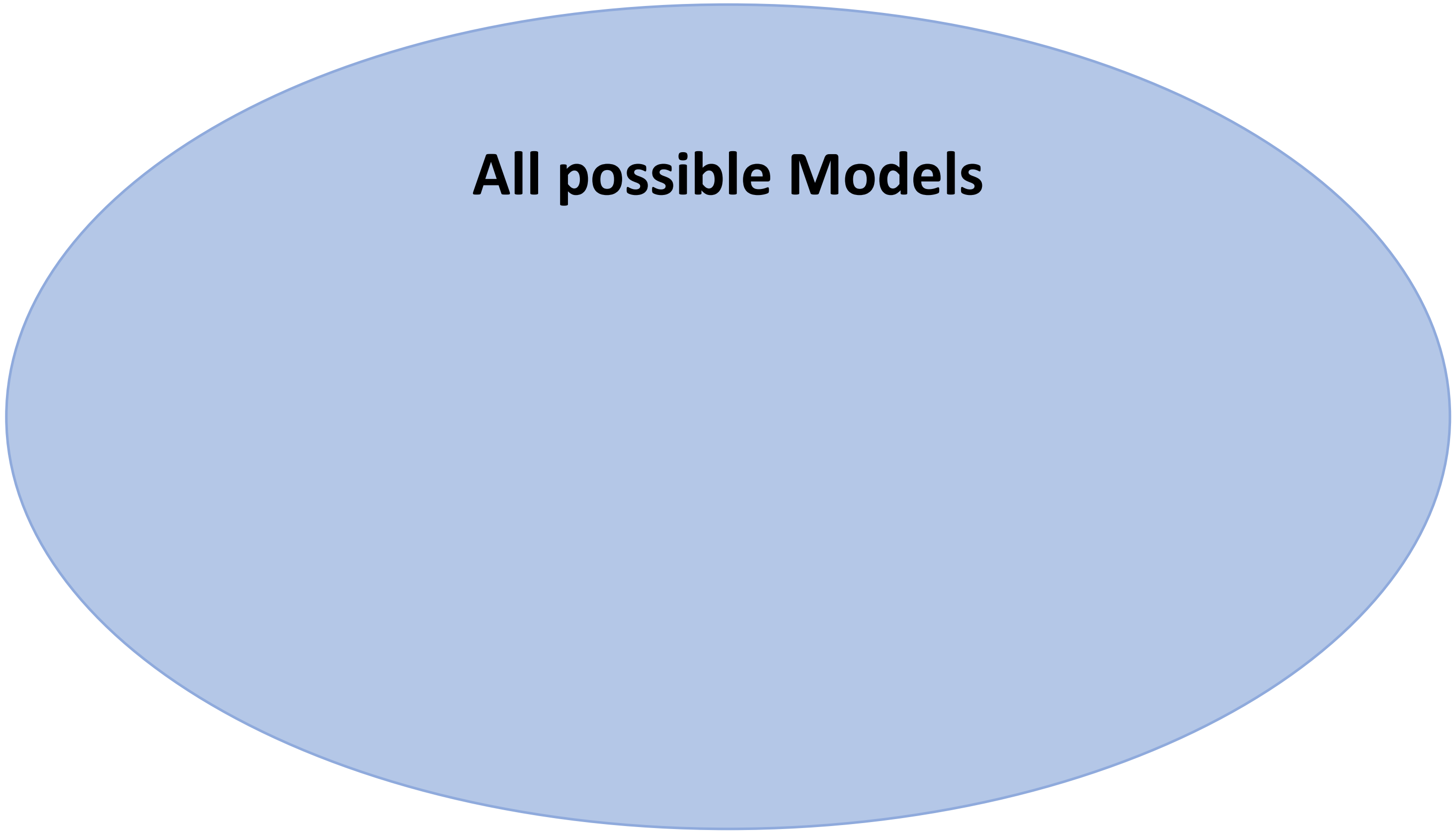


- All Models



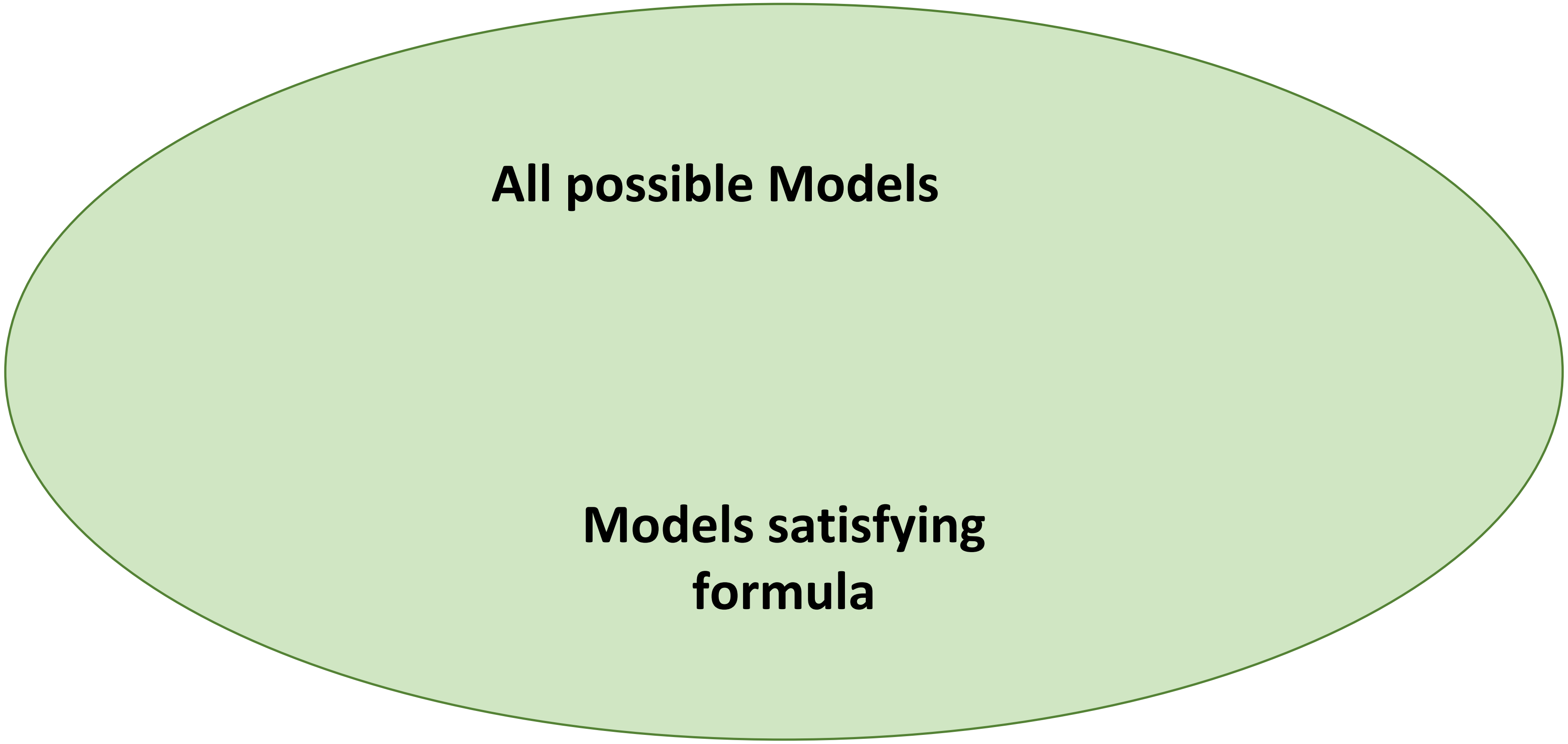
Unsatisfiability - UNSAT - Contradiction

- A formula that is not satisfiable



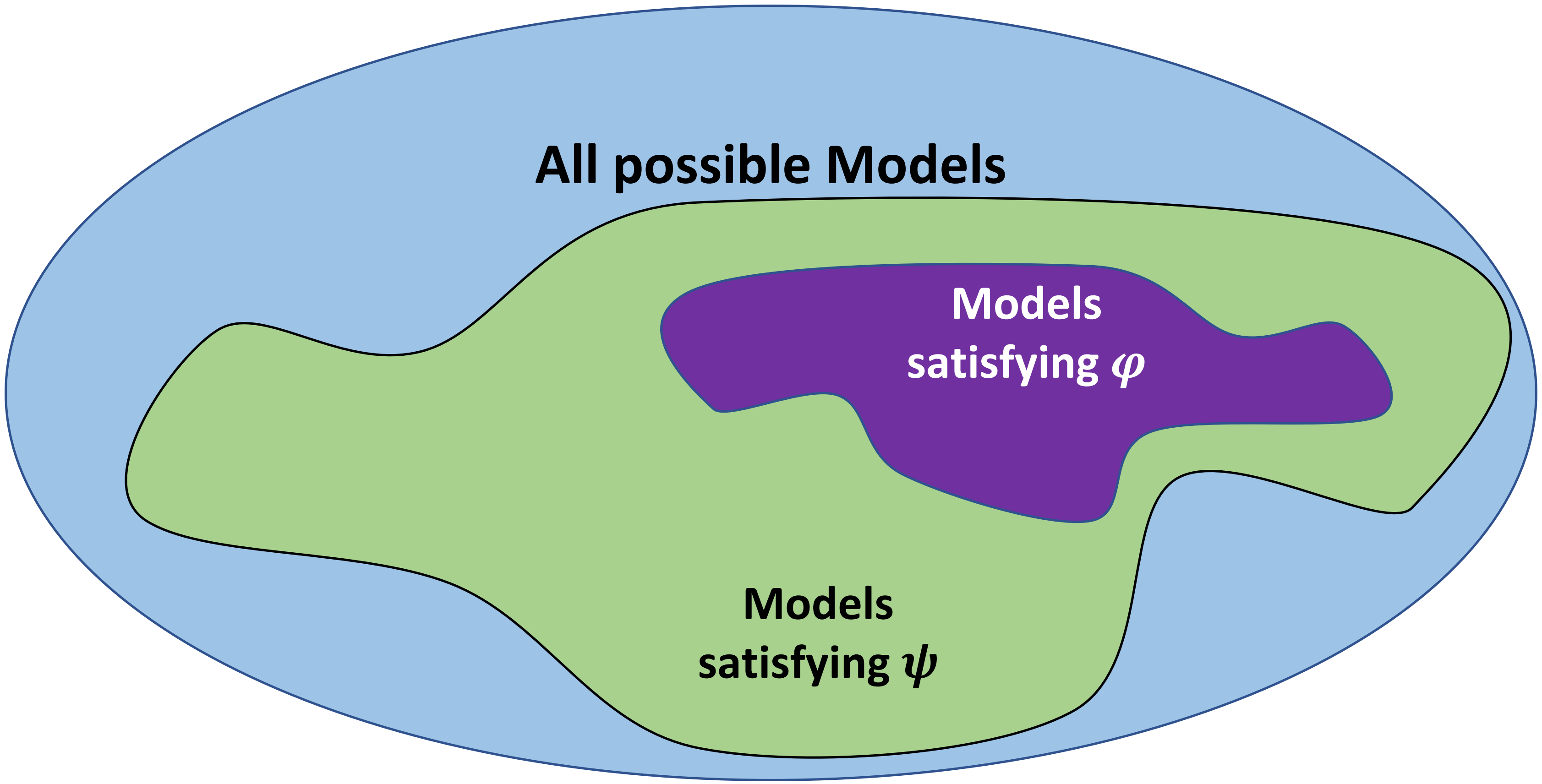
Validity - Tautology

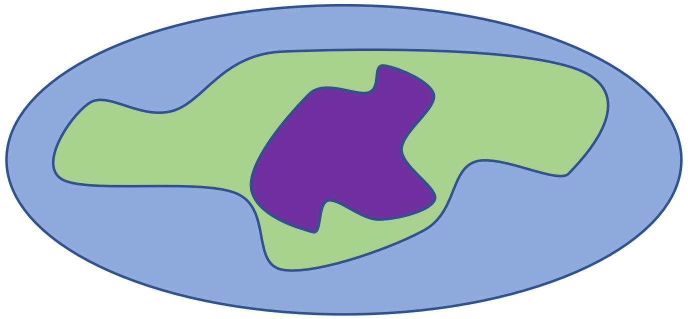
- All models satisfy formula



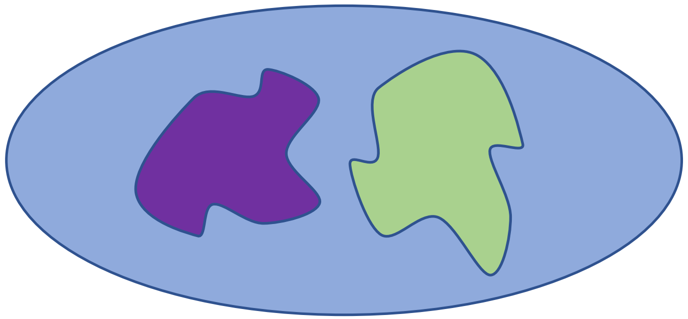
Semantic Entailment $\varphi \vDash \psi$

- φ is a special case of ψ

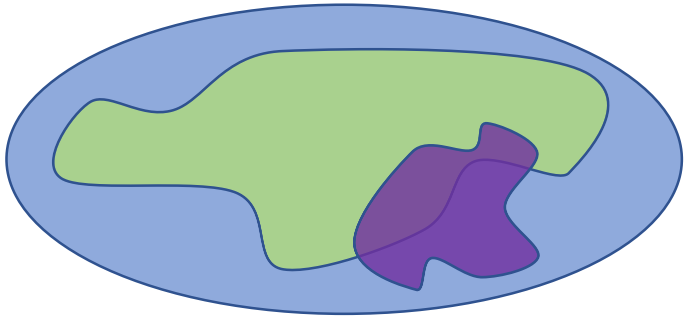




$\varphi \vDash \psi$

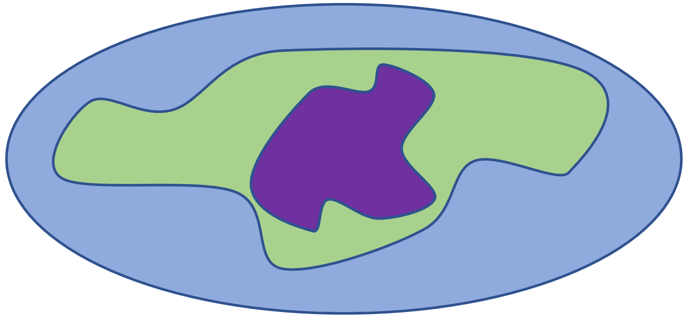


$\varphi \nvDash \psi$



$\varphi \nvDash \psi$

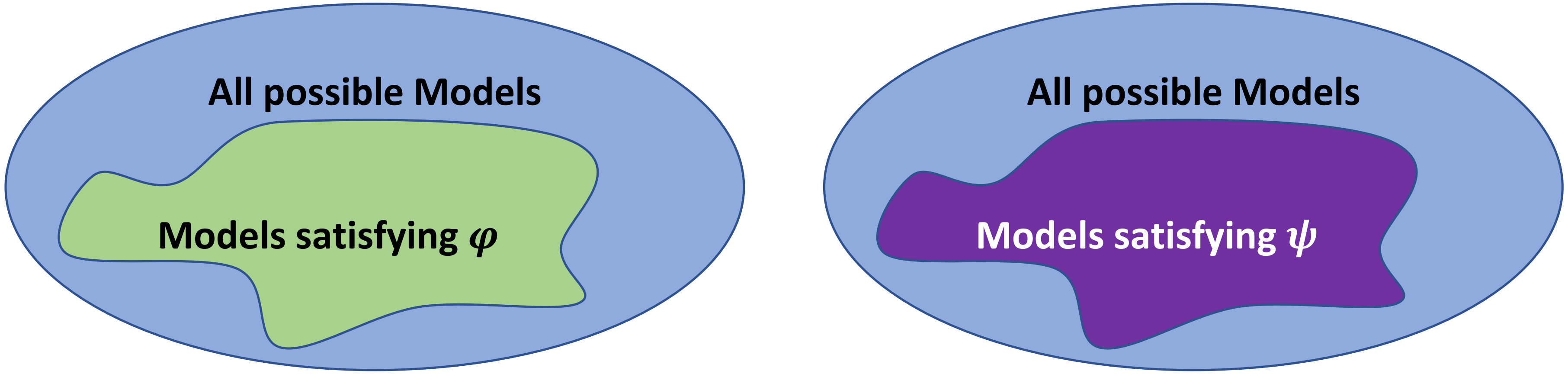
Semantic Entailment $\varphi \vDash \psi$



- Written: $\varphi \vDash \psi$ (Latex: \models)
- Meaning: $\mathcal{M} \vDash \varphi \quad \Rightarrow \quad \mathcal{M} \vDash \psi$
- Examples:
  - $(p \wedge q) \vDash p$
  - $(p \vee q) \nvDash p$

Semantic Equivalence $\varphi \equiv \psi$

- Special Case of Semantic Entailment
  - $\varphi \equiv \psi$ means that φ, ψ are satisfied by the same models.
  - Thus $\varphi \vDash \psi$, and $\psi \vDash \varphi$.



Truth Tables

- Used to check for *validity*, *satisfiability*, *semantic entailment* or *equivalence*
- Row for each Model $\mathcal{M}_i$
  - $\#Rows = 2^{\#Vars}$
- Column for each sub-formula φ
- Entry E_{ij}
  - True, if $\mathcal{M}_i \models \varphi_j$
  - False, if $\mathcal{M}_i \not\models \varphi_j$

Truth Tables

- Used to check for *validity*, *satisfiability*, *semantic entailment* or *equivalence*

- Row for each Model $\mathcal{M}_i$
  - $\#Rows = 2^{\#Vars}$ ⇐ Huge disadvantage of truth tables

- Column for each sub-formula φ

- Entry E_{ij}
  - True, if $\mathcal{M}_i \vDash \varphi_j$
  - False, if $\mathcal{M}_i \nvDash \varphi_j$

Truth Tables Usage

- Satisfiability
  - At least one row with **True**?
- Validity
  - All rows **True**?
- Entailment $\varphi \vDash \psi$
  - ψ has **True** at least where φ has **True**?
- Equivalence $\varphi \equiv \psi$
  - φ, ψ have **True** in same rows?

Outline

- Declarative Sentences ✔
- Syntax ✔
  - Symbols & Grammar
  - Parse Tree
- Semantics ✔
  - Meaning
  - Models
  - Truth Tables
  - Validity, Satisfiability, Entailment & Equivalence
- **Examples**

Formulate sentences with prop. logic

- Two straight lines can either be parallel or intersecting, but not both.
- Tomorrow is Tuesday, if and only if today is Monday.
- Students have to present their solutions at the blackboard.
- If I get a bike and if the weather is fine then I go by bike to work.

Formulate sentences with prop. logic

- Two straight lines can either be parallel or intersecting, but not both
  - p: “lines are parallel” q: “lines are intersecting”
  - $(p \vee q) \wedge \neg(p \wedge q)$
- Tomorrow is Tuesday, if and only if today is Monday.
- Students have to present their solutions at the blackboard.
- If I get a bike and if the weather is fine then I go by bike to work.

Formulate sentences with prop. logic

- Two straight lines can either be parallel or intersecting, but not both
  - p: “lines are parallel” q: “lines are intersecting”
  - $(p \vee q) \wedge \neg(p \wedge q)$
- **Tomorrow is Tuesday**, if and only if **today is Monday**.
  - p: „today is Monday“, q: tomorrow is Tuesday
  - $(p \leftrightarrow q) \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
- Students have to present their solutions at the blackboard.
- If I get a bike and if the weather is fine then I go by bike to work.

Formulate sentences with prop. logic

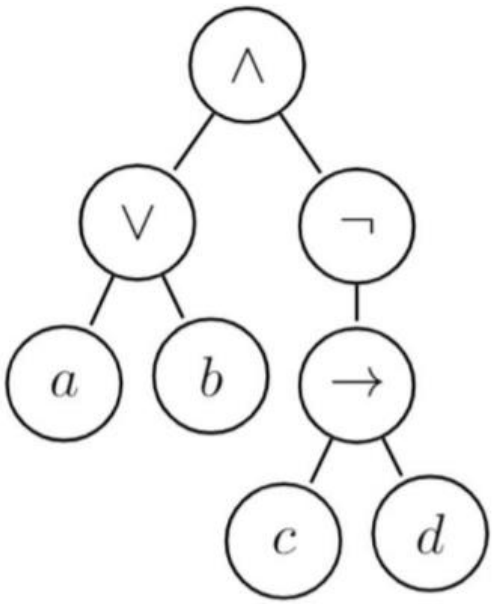
- Two straight lines can either be parallel or intersecting, but not both
  - p: “lines are parallel” q: “lines are intersecting”
  - $(p \vee q) \wedge \neg(p \wedge q)$
- **Tomorrow is Tuesday**, if and only if **today is Monday**.
  - p: „today is Monday“, q: tomorrow is Tuesday
  - $(p \leftrightarrow q) \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
- Students have to present their solutions at the blackboard.
  - q: “Students have to present their solutions at the blackboard”
  - q
- If I get a bike and if the weather is fine then I go by bike to work.

Formulate sentences with prop. logic

- Two straight lines can either be parallel or intersecting, but not both
  - p: “lines are parallel” q: “lines are intersecting”
  - $(p \lor q) \land \neg(p \land q)$
- Tomorrow is Tuesday, if and only if today is Monday.
  - p: „today is Monday“, q: tomorrow is Tuesday
  - $(p \leftrightarrow q) \equiv (p \rightarrow q) \land (q \rightarrow p)$
- Students have to present their solutions at the blackboard.
  - q: “Students have to present their solutions at the blackboard”
  - q
- If I get a bike and if the weather is fine then I go by bike to work.
  - p: “I get a bike.” q: “The weather is fine.” r: “I go by bike to work.”
  - $p \rightarrow (q \rightarrow r)$

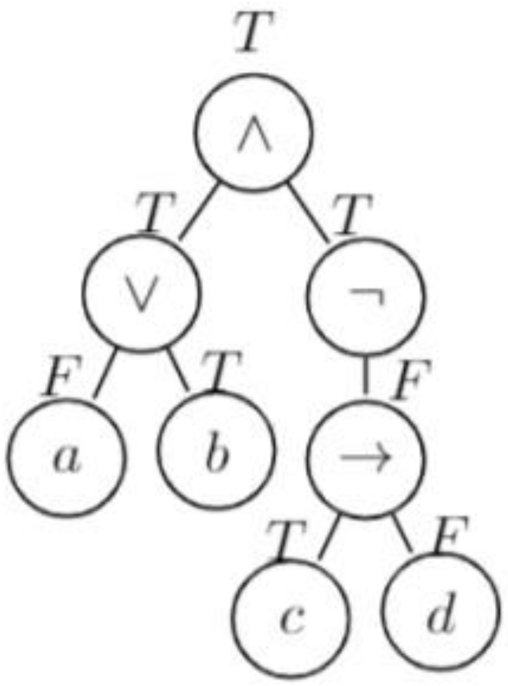
Draw parse tree for φ and evaluate φ under $\mathcal{M}_1$

- $\varphi = (a \vee b) \wedge (\neg(c \rightarrow d))$
- $\mathcal{M}_1 : a = F,\ b = T,\ c = T,\ d = F$



Draw parse tree for φ and evaluate φ under $\mathcal{M}_1$

- $\varphi = (a \lor b) \land (\neg(c \to d))$
- $\mathcal{M}_1 : a = F,\ b = T,\ c = T,\ d = F$

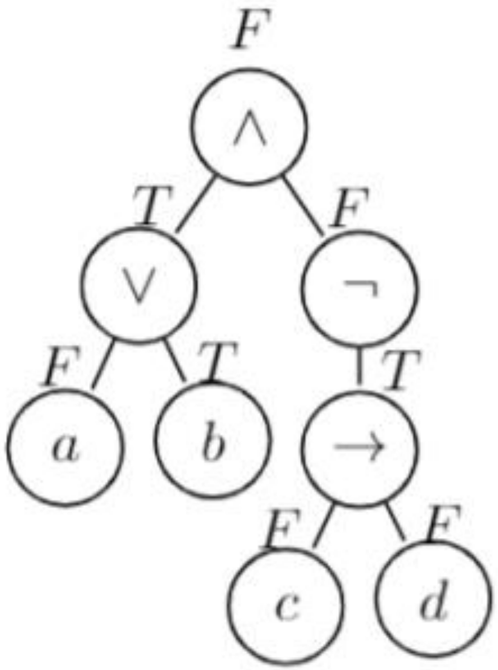


$\varphi^{\mathcal{M}_1} = T$

$M_1 \vDash \varphi$

Draw parse tree for φ and evaluate φ under $\mathcal{M}_2$

- $\varphi = (a \vee b) \wedge (\neg (c \rightarrow d))$
- $\mathcal{M}_2 : a = F,\ b = T,\ c = F,\ d = F$



$\varphi^{\mathcal{M}_2} = F$

$M_2 \nvDash \varphi$

Usage of Truth Table: $\varphi = a \wedge \neg(b \rightarrow c)$; $\psi = \neg a$

- Draw a truth table
- Answer the following questions:
  a. Is φ Satisfiability?
  b. Is φ valid?
  c. Does ψ semantically entail φ?
  d. Does φ semantically entail ψ?

Usage of Truth Table: $\varphi = a \land \neg(b \to c)$; $\psi = \neg a$

- Draw a truth table
- Answer the following questions:
  a. Is φ Satisfiability?
  b. Is φ valid?
  c. Does ψ semantically entail φ?
  d. Does φ semantically entail ψ?

a	b	c	$b \to c$	$\neg(b \to c)$	φ	ψ
F	F	F	*T*	*F*	*F*	T
F	F	T	*T*	*F*	*F*	T
F	T	F	*F*	*T*	*F*	T
F	T	T	*T*	*F*	*F*	T
T	F	F	*T*	*F*	*F*	F
T	F	T	*T*	*F*	*F*	F
T	T	F	*F*	*T*	*T*	F
T	T	T	*T*	*F*	*F*	F

Usage of Truth Table: $\varphi = a \wedge \neg(b \rightarrow c)$; $\psi = \neg a$

- Draw a truth table
- Answer the following questions:
  a. Is φ Satisfiability?
  b. Is φ valid?
  c. Does ψ semantically entail φ?
  d. Does φ semantically entail ψ?

a	b	c	$b \rightarrow c$	$\neg(b \rightarrow c)$	φ	ψ
F	F	F	T	F	F	T
F	F	T	T	F	F	T
F	T	F	F	T	F	T
F	T	T	T	F	F	T
T	F	F	T	F	F	F
T	F	T	T	F	F	F
T	T	F	F	T	T	F
T	T	T	T	F	F	F

Solution:

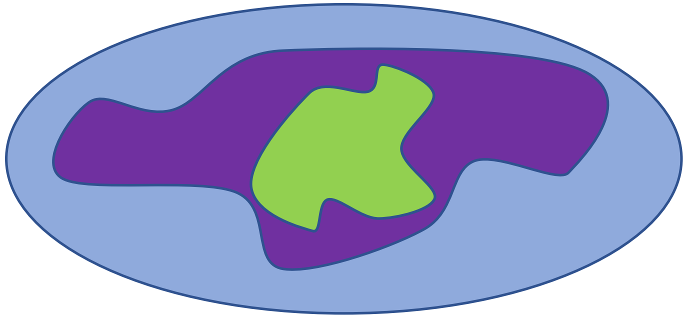
a. Yes

b. No

Usage of Truth Table: $\varphi = a \wedge \neg(b \rightarrow c)$; $\psi = \neg a$

- Draw a truth table
- Answer the following questions:
  - a. Is φ Satisfiability?
  - b. Is φ valid?
  - c. Does ψ semantically entail φ?
  - d. Does φ semantically entail ψ?

a	b	c	$b \rightarrow c$	$\neg(b \rightarrow c)$	φ	ψ
F	F	F	T	F	F	T
F	F	T	T	F	F	T
F	T	F	F	T	F	T
F	T	T	T	F	F	T
T	F	F	T	F	F	F
T	F	T	T	F	F	F
T	T	F	F	T	T	F
T	T	T	T	F	F	F



$\psi \vDash \varphi$

$\mathcal{M} \vDash \psi \quad \Rightarrow \quad \mathcal{M} \vDash \varphi$

Solution:

a. **Yes**

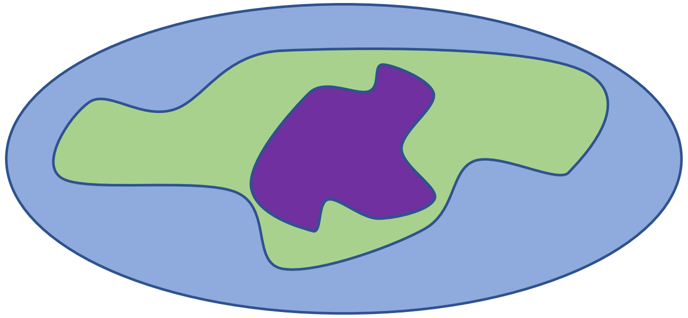
b. **No**

c. **No**

Usage of Truth Table: $\varphi = a \wedge \neg(b \rightarrow c)$; $\psi = \neg a$

- Draw a truth table
- Answer the following questions:
  a. Is φ Satisfiability?
  b. Is φ valid?
  c. Does ψ semantically entail φ?
  d. Does φ semantically entail ψ?

a	b	c	$b \rightarrow c$	$\neg(b \rightarrow c)$	φ	ψ
F	F	F	T	F	F	T
F	F	T	T	F	F	T
F	T	F	F	T	F	T
F	T	T	T	F	F	T
T	F	F	T	F	F	F
T	F	T	T	F	F	F
T	T	F	F	T	T	F
T	T	T	T	F	F	F



$\varphi \vDash \psi$

$\mathcal{M} \vDash \varphi \quad \Rightarrow \quad \mathcal{M} \vDash \psi$

Solution:

a. Yes
b. No
c. No
d. No

Learning Targets

- Syntax
  - Explain syntax of prop. formulas
  - Draw parse tree of prop. formulas
- Semantics
  - Model sentences as prop. formula
  - Explain semantics of prop. Formulas
  - Explain what models are
  - Construct and use truth tables
  - Explain and decide validity, satisfiability, semantic entailment and equivalence
    - Using truth tables

Thank You