Predicate Logic

(aka. First-Order Logic)

Syntax, Semantics, Models

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Motivation

- Limits of Propositional Logic
  - Need for more expressive language
- Syntax
  - Construct formulas
- Semantics, Models
  - Understand formulas

Outline

- Intuitive Example
- Syntax
- Models
- Semantics

Intuitive Example

Rex

Rex is a dog. (p: $Dog(Rex)$)   All dogs have fur. (q: $\forall x\,(Dog(x) \rightarrow Fur(x))$)

Rex has fur. (r: $Fur(Rex)$)

•) $p, q \vdash r$ $p = T,\ q = T,\ r = \bot$ ↯

•) $Dog(Rex),\ \forall x\,(Dog(x) \rightarrow Fur(x)) \vdash Fur(Rex)$ ✓

More Examples

- Not all birds can fly.

  $\neg \forall x (Bird(x) \rightarrow Fly(x))$

- Every student is younger than some teacher.

  $\forall x \exists y (Student(x) \wedge Teacher(y) \rightarrow Younger(x, y))$

- Andi and Paul have the same maternal grandmother.

  $mother(mother(Andi)) = mother(mother(Paul))$

More Examples

- Some people in class visited the Grand Canyon.

$$\exists x\, (Person(x) \wedge InClass(x) \wedge VisitedGC(x))$$

- All integers are either even or odd.

$$\forall x\, (Integer(x) \rightarrow (Even(x) \oplus Odd(x)))$$

More Examples

- „Alice has no sister.“
  - Alice: constant
  - Sister(x,y)… x is y's sister
  - $\neg\exists x\ Sister\ (Alice, x)$

- „A person who wears a crown is either a king or a queen.“
  - P(x)… x is a person C(x)… x wears a crown
  - K(x)… x is a king Q(x)…. x is a queen
  - $\forall x((P(x) \land C(x)) \rightarrow (K(x) \oplus Q(x)))$

Syntax of Predicate Logic

- Subjects (Objects in OOP)
  - Rex
  - Andy
  - Paul
  - …

Terms

- Truth values
  - Dog (Rex)
  - …

Formulas

From Terms to Formulas

Formula

$$\overbrace{\underbrace{P}_{\text{Predicate}}(\underbrace{x}_{\text{Term}}, \underbrace{f(y)}_{\text{Term}})}^{\text{Formula}}$$

Predicate **Term** **Term**

Syntax of Predicate Logic

- Variables $\mathbb{V}$
  - $x, y, z, \dots$
- Functions $\mathbb{F}$
  - $f, g, h, \dots$ (arity > 0)
  - constants (arity = 0)
- Terms $\mathbb{T}$
  - defined recursively
- Predicates $\mathbb{P}$
  - $P, Q, R, \dots$ (arity > 0)
  - Prop. constants (arity = 0)

Terms $\mathbb{T}$

- Variable
- Nullary Function (constant)
- Terms $t_1, t_2, \dots, t_n$, n-ary Function f

$$\Updownarrow$$

$f(t_1, t_2, \dots, t_n)$ is a term

Formulas

Preconditions:

- Terms $t_1, t_2, \dots, t_n$
- n-ary predicate symbol P
- formulas ϕ, ψ
- Variable x

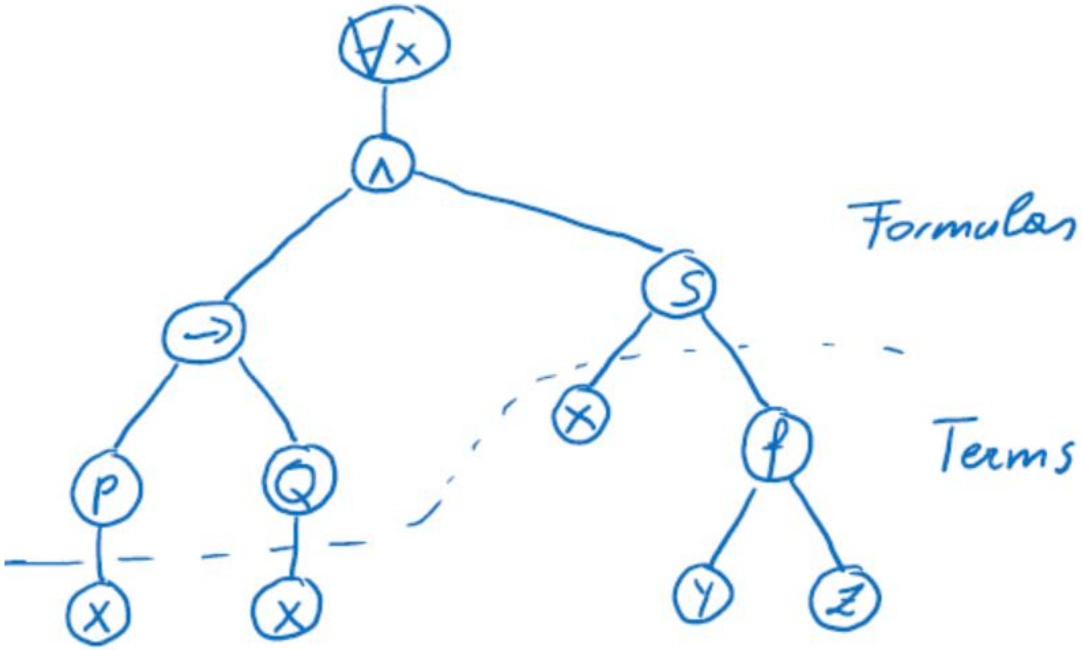
- $P(t_1, t_2, \dots, t_n)$
- $\neg\phi$
- $\phi \land \psi$, $\phi \lor \psi$, $\phi \rightarrow \psi$
- $\forall x\, \phi$
- $\exists x\, \phi$

Binding Priorities

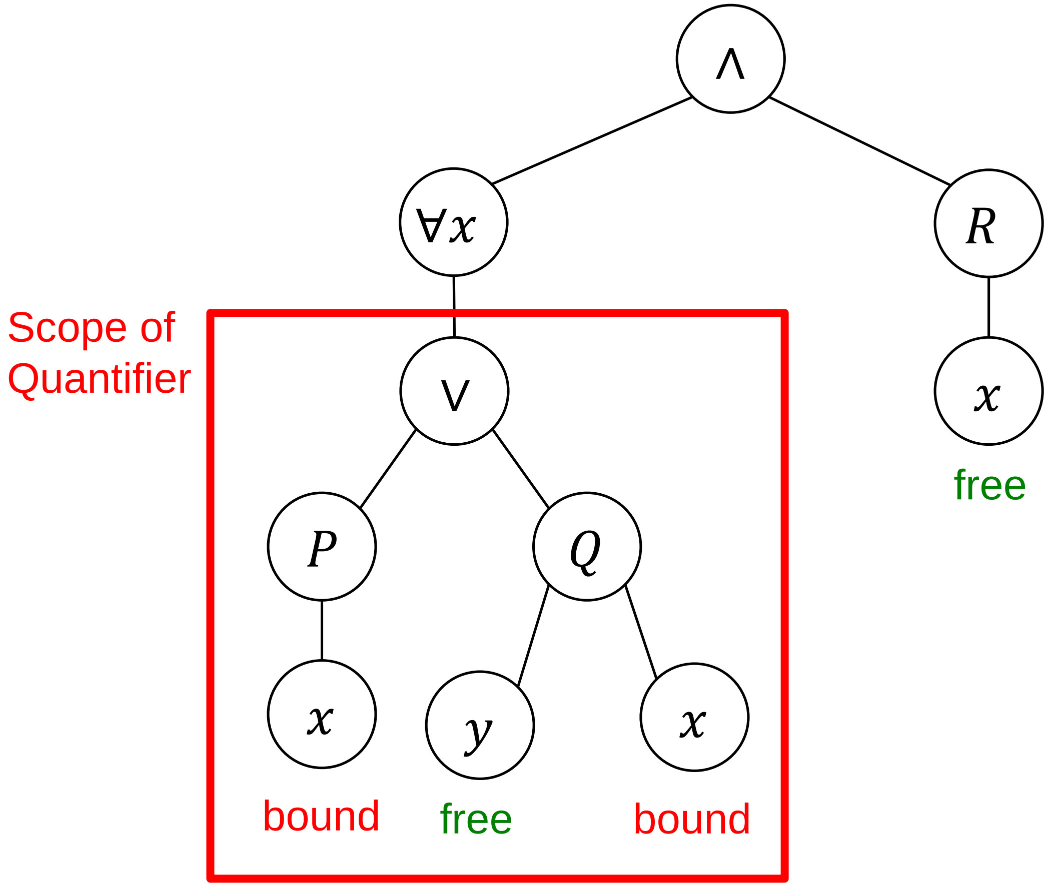
1. $\forall$, $\exists$, $\neg$ e.g., $\forall x\Big(\big(P(x) \rightarrow Q(x)\big) \wedge S(x, f(y,z))\Big)$
2. $\wedge$
3. $\vee$
4. $\rightarrow$
   - right-associative

Syntax Tree

$$\forall x\Big(\big(P(x) \to Q(x)\big) \land S(x, f(y,z))\Big)$$



Free and Bound Variables



Model M for $(\mathbb{F}, \mathbb{P})$

- Non-empty set A
  - "**Universe** of Values" or "**Domain** of Values"
  - Possibly infinite
- For nullary $f \in \mathbb{F}$: concrete element $f^M \in A$
- For nullary $P \in \mathbb{P}$: true or false
- For other $f \in \mathbb{F}$: concrete function $f^M: A^n \to A$
  - Defined by e.g. function table
- For $P \in \mathbb{P}$ with arity $n > 0$: subset $P^M \subseteq A^n$
  - Tuples which make P true
- Look-Up Table for Free Variables
  - Function $l: \mathbb{V} \to A$

Semantics of Predicate Logic

- Formula ϕ
  - Over $\mathbb{F}$, $\mathbb{P}$

- Model M
  - Domain A, For $\mathbb{F}$, $\mathbb{P}$, Lookup Table l for $\mathbb{V}$

- $M \vDash \phi$?

Semantics of Predicate Logic

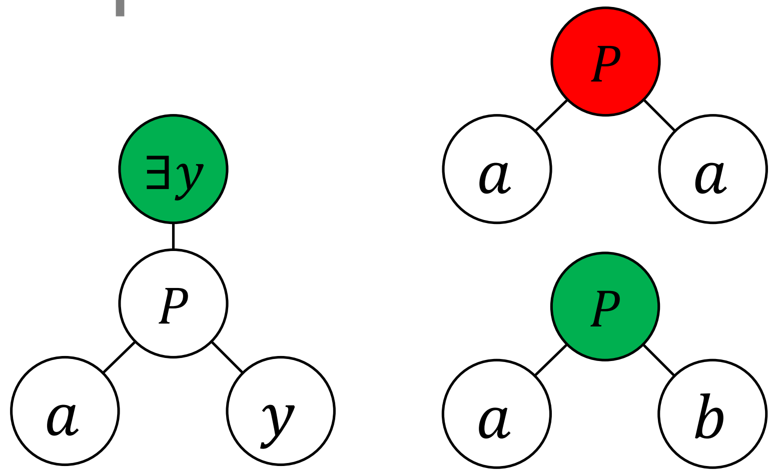
- For ϕ of the form $P(t_1, t_2, \ldots, t_n)$
  - Interpret $t_1, \ldots, t_n$ via l, M
  - Obtain $(a_1, a_2, \ldots, a_n)$
  - $M \vDash P(t_1, t_2, \ldots, t_n)$ iff $(a_1, a_2, \ldots, a_n) \in P^M$

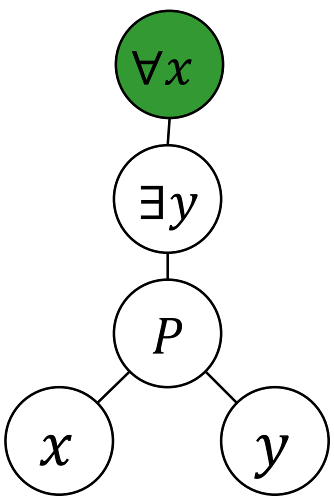
Semantics of Predicate Logic

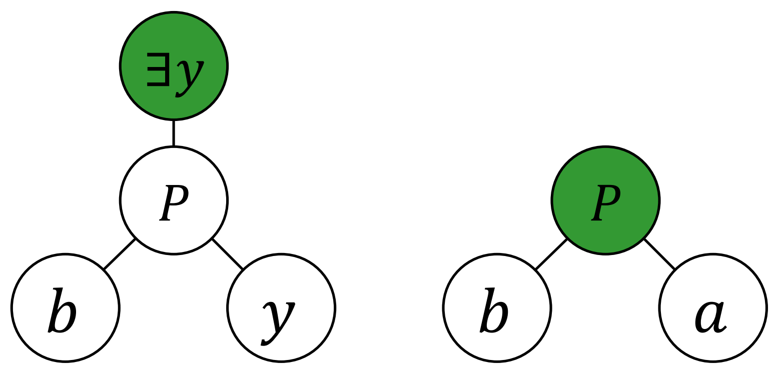
- For ϕ of the form $\forall x\ \psi$
  - $M \vDash \forall x\ \psi$ iff $M \vDash \psi$, for all $a \in A$
- For ϕ of the form $\exists x\ \psi$
  - $M \vDash \exists x\ \psi$ iff $M \vDash \psi$, for at least one $a \in A$
- For ϕ of the form $\neg\psi$, $\psi_1 \wedge \psi_2$, $\psi_1 \vee \psi_2$, $\psi_1 \rightarrow \psi_2$
  - Like in propositional logic

Example I

- $\phi = \forall x \exists y. P(x, y)$
- M:
  - $A = \{a, b\}$
  - $P^M = \{(a, b), (b, a)\}$

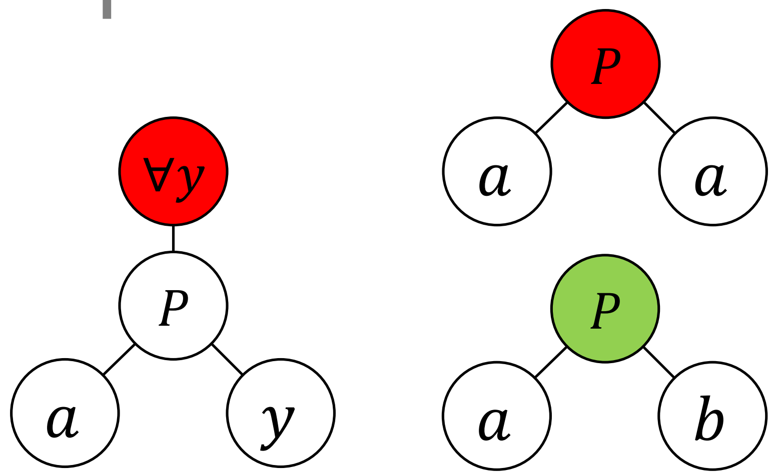


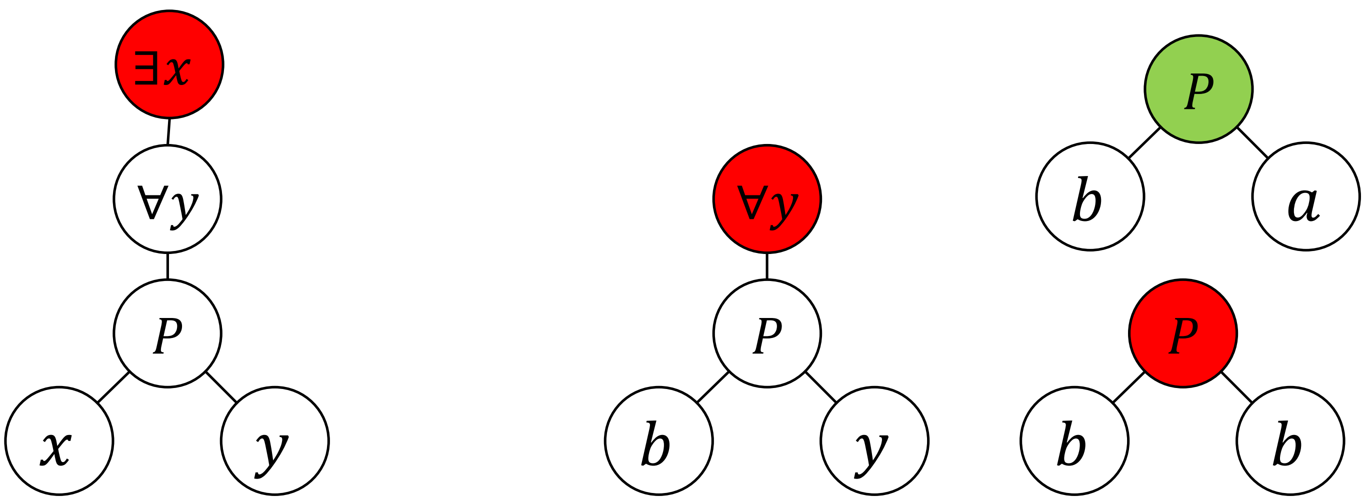




Example II

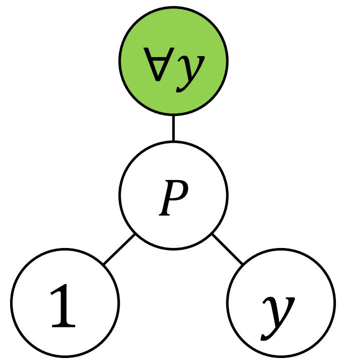
- $\psi = \exists x \forall y . P(x, y)$
- M:
  - $A = \{a, b\}$
  - $P^M = \{(a, b), (b, a)\}$

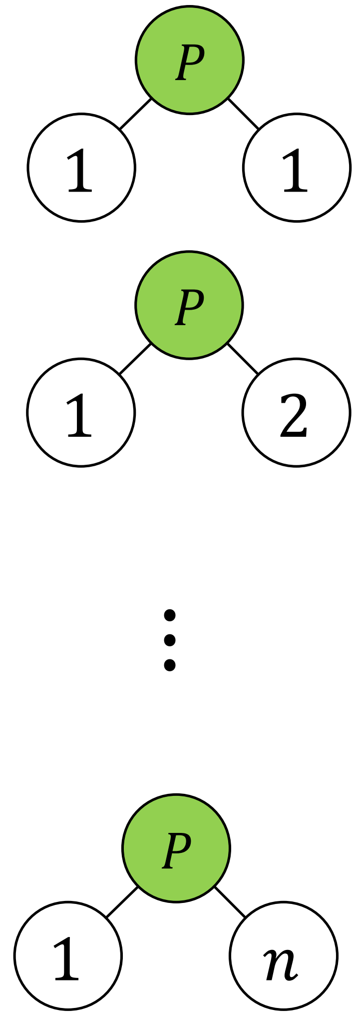


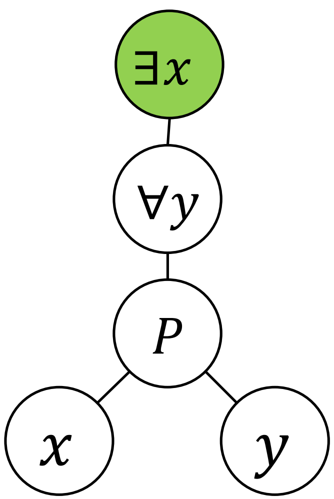


Example III

- $\psi = \exists x \forall y. P(x, y)$
- M:
  - $A = \mathbb{N}$
  - $P^M = \leq = \{(1,1), (1,2), \ldots, (2,2), \ldots\}$







Semantic Entailment

- $\phi_1, \phi_2, \phi_3, \ldots \vDash \psi$ iff:
  - For all models M
  - $M \vDash \phi$ for all $\phi_1, \phi_2, \phi_3, \ldots$, then also $M \vDash \psi$

Validity

- ψ is **valid** iff:
  - For all models M
  - $M \vDash \psi$

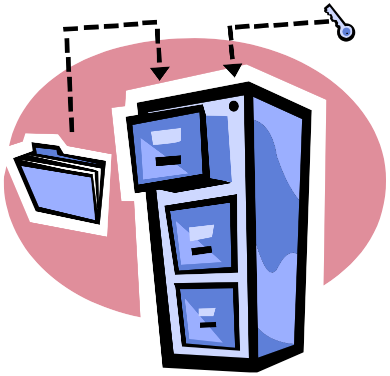
Satisfiability

- ψ is **satisfiable** iff:
  - One model M
  - $M \vDash \psi$

Semantics of Equality

- Special case “=“
  - Denotes “real” equality
    - $=^M = \{(a_1, a_1),\ (a_2, a_2),\ (a_3, a_3), \ldots\}$
  - In ***all*** models

Summary



- Syntax
  - Predicates
  - Functions
  - Quantifiers
- Semantics
  - Models
  - How to evaluate
- Validity, Satisfiability