# Questionnaire “Logic and Computability”

# Summer Term 2022

## Contents

# 2 Natural deduction for Propositional Logic

## 2.1 Lecture

For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

For proofs, clearly indicate which rule, and what assumptions/premises/
intermediate results you are using in each step. Also clearly indicate the scope of any boxes you use.

For counterexamples, give a complete model. Show that the model satisfies the premise(s) of the sequent in question, but does not satisfy the respective conclusion.

### 2.1.1 Rules for natural deduction

1. [Lecture] Give the definition of a sequent. Give an example of a sequent and name the parts the sequent consists of.
2. [Lecture] Look at the following statements and tick them if they are true.
   - ☐ In a sequent, premises entail a conclusion.
   - ☐ In a sequent, conclusions entail a premise.
   - ☐ A sequent is valid, if no proof for it can be found.
   - ☐ A sequent is valid, if a proof for it can be found.
3. [Lecture] State the *AND-introduction* rule ($\wedge$i). Explain how the rule works.
4. [Lecture] $p, q, r \;\vdash\; p \wedge (q \wedge r)$
5. [Lecture] $p \wedge (q \wedge r) \;\vdash\; q$
6. [Lecture] $p \wedge q, \neg q \wedge r \;\vdash\; \neg\neg p \wedge \neg\neg r$
7. [Lecture] $\neg\neg\neg p \wedge q, \neg\neg r \;\vdash\; r \wedge \neg p \wedge \neg\neg q$
8. [Lecture] $p \wedge q, q \to \neg\neg r \;\vdash\; p \wedge r$
9. [Lecture] Explain the *implication-elimination* rule ($\to$e). Show how the *Modus Tollens* rule derives from the $\to$e rule?
10. [Lecture] $\neg p \to q, \neg\neg\neg q \wedge r \;\vdash\; p \wedge \neg\neg\neg q$
11. [Lecture] Translate the following reasoning into a sequent. If the sequent is valid, proof it using the rules of natural deduction. If the sequent is not valid, provide a counter example.

    If I press the button, the window opens.
    I pressed the button.
    Therefore, the window is open.

12. [Lecture] Explain the concept of boxes in deduction rules and why they are needed. What does it mean if you make an *assumption* within a box? Where is this assumption valid?
13. [Lecture] $p \to (q \wedge r), q \to s \;\vdash\; p \to (s \wedge r)$
14. [Lecture] Why are there two rules for the $\vee$-*introduction* rule. Explain, why you are able to connect any formula to a certain formula $\varphi$ using the connective $\vee$.
15. [Lecture] $p \wedge q, r \to s \;\vdash\; (p \vee (r \to s)) \wedge (q \vee ((t \vee r) \to u))$

16. [Lecture] Explain the *OR-elimination* ($\vee$-$e$) rule of the natural deduction calculus. In particular, why does it rule require two boxes?
17. [Lecture] $p \vee \neg\neg q, \neg p \wedge \neg q \vdash s \vee \neg t$
18. [Lecture] $\neg q \vee \neg p \vdash \neg(q \wedge p)$
19. [Lecture] $\vdash ((p \to q) \to p) \to p$
20. [Lecture] $\neg(q \wedge p) \vdash \neg q \vee \neg p$
21. [Lecture] Explain in your own words, how to proof a sequent in the natural deduction calculus. What steps do you need to take and which tips can be helpful when solving such proofs?

#### 2.1.2 Soundness and completeness of natural deduction

22. [Lecture] "Natural deduction for propositional logic is *sound* and *complete*." Explain in your own words what this means.
23. [Lecture] How can you show that a sequent is not valid? Is this a consequence of soundness or completeness. Explain your answer.
24. [Lecture] $p \to q, q \to r \vdash r$.
25. [Lecture] Translate the following reasoning into a sequent. If the sequent is valid, proof it using the rules of natural deduction. If the sequent is not valid, provide a counter example.

    If I press the button, the window opens.
    The window is open.
    Therefore, I pressed the button.

## 2.2 Practicals

For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

For proofs, clearly indicate which rule, and what assumptions/premises/
intermediate results you are using in each step. Also clearly indicate the scope of any boxes you use.

For counterexamples, give a complete model. Show that the model satisfies the premise(s) of the sequent in question, but does not satisfy the respective conclusion. For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

1. [Practicals] **[2 Points]**

   (a) If I am ill, I go to the doctor.
   I am ill.
   Therefore, I go to the doctor.

   (b) If I am ill, I go to the doctor.
   I go to the doctor.
   Therefore, I am ill.

(c) (Solve without using the Modus Tollens)
If I am ill, I go to the doctor.
I did not go to the doctor.
Therefore, I am not ill.

2. [Practicals] **[2 Points]**
   (a) $(p \wedge q) \wedge \neg r \;\vdash\; q \vee r$
   (b) $(p \vee q) \wedge \neg r \;\vdash\; q \wedge r$
3. [Practicals] **[2 Points]**
   (a) $\vdash\; (p \to q) \to p$
   (b) $\vdash\; p \to (q \to p)$
4. [Practicals] **[2 Points]** $\neg(a \wedge b) \vee \neg c \;\vdash\; \neg(a \wedge b) \to c \vee a$
5. [Practicals] **[2 Points]** $p \wedge q \vee r \;\vdash\; (p \vee r) \wedge (q \vee r)$
6. [Practicals] **[2 Points]** $\neg\neg x \to \neg y \wedge z \;\vdash\; z \to \neg x \wedge \neg\neg y$
7. [Practicals] **[2 Points]** $\vdash\; \neg(p \wedge q) \vee p$
8. [Practicals] **[2 Points]** $\neg(a \vee b) \;\vdash\; \neg a \wedge \neg b$
9. [Practicals] **[2 Points]** $(s \vee \neg u) \to t \;\vdash\; (\neg s \wedge u) \vee t$
10. [Practicals] **[2 Points]** $\neg\neg k \to (l \vee m), \neg\neg\neg l \to m \;\vdash\; \neg k \vee (l \vee \neg\neg m)$

## 2.3 Self Evaluation

For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

For proofs, clearly indicate which rule, and what assumptions/premises/intermediate results you are using in each step. Also clearly indicate the scope of any boxes you use.

For counterexamples, give a complete model. Show that the model satisfies the premise(s) of the sequent in question, but does not satisfy the respective conclusion.

### 2.3.1 Rules for natural deduction

25. [Self-Assessment] $p \wedge (q \wedge r) \;\vdash\; (p \wedge q) \wedge r$
26. [Self-Assessment] $\neg\neg p \wedge \neg\neg q, r \wedge s \;\vdash\; (p \wedge r) \wedge \neg\neg s$
27. [Self-Assessment] $(p \to q) \wedge (q \to r), p \;\vdash\; \neg\neg r \wedge \neg p$
28. [Self-Assessment] $(\neg p \to q) \wedge (q \to r), \neg r \;\vdash\; \neg\neg\neg r \wedge \neg p$
29. [Self-Assessment] Explain the *implication-introduction* rule ($\to$i).
30. [Self-Assessment] $(p \to q) \to r \;\vdash\; \neg r \wedge \neg s \to \neg(p \to q)$
31. [Self-Assessment] $p \to q \;\vdash\; (r \to p) \to (r \to q)$
32. [Self-Assessment] $p \to q, p \wedge (r \vee q) \;\vdash\; (q \to p) \to ((s \wedge t) \vee q) \wedge (r \vee q)$
33. [Self-Assessment] $p \vee q, \neg p \vee r \;\vdash\; q \vee r$

34. [Self-Assessment] $p \to q, p \land r \lor q \;\vdash\; (q \to p) \to ((s \land t) \lor q) \land (r \lor q)$

35. [Self-Assessment] $p \lor q, p \to r, \neg s \to \neg q \;\vdash\; r \lor s$

36. [Self-Assessment] Look at the following statements and tick them if they are true.

    - ☐ Given two premises $\varphi$ and $\psi$, we can conclude that $\varphi \land \psi$ holds using $\land$-introduction.
    - ☐ Given two premises $\varphi$ and $\psi$, we can conclude that $\varphi \lor \psi$ holds using $\lor$-introduction.
    - ☐ Given a premise $\varphi \land \psi$, we can conclude $\varphi$ with $\land$-elimination.
    - ☐ Given a premise $\varphi \lor \psi$, we can conclude $\varphi$ with $\lor$-elimination.

37. [Self-Assessment] $\vdash\; p \to (q \to p)$

38. [Self-Assessment] Translate the following reasoning into a sequent. If the sequent is valid, proof it using the rules of natural deduction. If the sequent is not valid, provide a counter example.

    If I press the button, the window opens.
    The window is not open.
    Therefore, I didn't press the button.

39. [Self-Assessment] Explain the $\bot$-*elimination* rule of the natural deduction calculus. Why can you deduce a formula $\varphi$ from something, that is wrong?

40. [Self-Assessment] $\neg q \lor p \;\vdash\; q \to (p \lor r)$

41. [Self-Assessment] $p \to (q \lor r), \neg q \land \neg r \;\vdash\; \neg p$

42. [Self-Assessment] Derive the *Proof-By-Contradiction*-rule from the $\neg$-*introduction* rule.

43. [Self-Assessment] $\neg(q \lor p) \;\vdash\; \neg q \land p$

44. [Self-Assessment] $\vdash\; (p \to q) \lor (q \to r)$

45. [Self-Assessment] $(p \to q) \land (q \to p) \;\vdash\; (p \land q) \lor (\neg p \land \neg q)$

46. [Self-Assessment] Look at the following statements and tick them if they are true.

    - ☐ A sequent always has to have at least one premise to be formally correct.
    - ☐ When proving a sequent you start your proof with the premise(s) and end it with the conclusion.
    - ☐ A natural deduction proof can theoretically have infinite assumption boxes in it.
    - ☐ A natural deduction rule can only be applied on the bottom-level connective of a formula.

### 2.3.2 Soundness and completeness of natural deduction

47. [Self-Assessment] Explain what it means that natural deduction for propositional logic is *sound*. What is the difference to *completeness*?

48. [Self-Assessment] Explain what it means that natural deduction for propositional logic is *sound*. What is the difference to *completeness*?

49. [Self-Assessment] Look at the following statements and tick them if they are true.

    - ☐ Any sequent that is a correct semantic entailment can be proven.

- ☐ Any sequent that can be proven is a correct semantic entailment.
- ☐ If a sequent is not provable, the semantic entailment relation does hold.
- ☐ If for a sequent the semantic entailment relation does not hold, it cannot be proven with natural deduction.

50. [Self-Assessment] Natural deduction for propositional logic is sound and complete. In the following list, mark each statement with either **S**, **C**, **B**, or **N**, depending on whether the corresponding statement follows from **S**oundness, **C**ompleteness, **B**oth, or **N**either. (Note: If a statement is in itself factually wrong, or has nothing to do with soundness and completeness, mark it **N**, since it follows from neither soundness nor completeness.)

- ☐ There is no correct sequent for which there is no proof.
- ☐ Every sequent has a proof.
- ☐ If all models that satisfy the premise(s) of a given sequent also satisfy the conclusion of the sequent, there exists a proof for the sequent.
- ☐ A sequent has a proof if and only if it is semantically correct.
- ☐ An incorrect sequent does not have a proof.
- ☐ Every propositional formula is either valid or not valid.
- ☐ If a model satisfies the premise(s) of a given sequent, but does not satisfy the conclusion of the sequent, it is not possible to construct a proof for the sequent.

51. [Self-Assessment] Natural deduction for propositional logic is sound and complete. In the following list, mark each statement with either **S**, **C**, **B**, or **N**, depending on whether the corresponding statement follows from **S**oundness, **C**ompleteness, **B**oth, or **N**either. (Note: If a statement is in itself factually wrong, or has nothing to do with soundness and completeness, mark it **N**, since it follows from neither soundness nor completeness.)

- ☐ A sequent has a proof if and only if it is semantically correct.
- ☐ If a model satisfies the premise(s) of a given sequent, but does not satisfy the conclusion of the sequent, it is not possible to construct a proof for the sequent.
- ☐ There is no correct sequent for which there is no proof.
- ☐ Every sequent has a proof.
- ☐ An incorrect sequent does not have a proof.
- ☐ Every propositional formula is either valid or not valid.
- ☐ If all models that satisfy the premise(s) of a given sequent also satisfy the conclusion of the sequent, there exists a proof for the sequent.

52. [Self-Assessment] Given an invalid sequent, how do you prove its invalidity?

53. [Self-Assessment] $\neg(p \vee \neg q) \vdash p$

54. [Self-Assessment] $p \rightarrow q \vdash ((p \vee q) \rightarrow p) \wedge (p \rightarrow (p \vee q))$

55. [Self-Assessment] $p \vee q, \neg q \vee r \vdash r$

56. [Self-Assessment] $(p \wedge q) \rightarrow (\neg r \wedge \neg s), \neg r \wedge \neg s \vdash p$