Questionnaire “Logic and Computability”

Summer Term 2022

Contents

7 Natural Deduction for Predicate Logic

7.1 Lecture

For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

For proofs, clearly indicate which rule, and what assumptions/premises/ intermediate results you are using in each step. Also clearly indicate the scope of any boxes you use.

For counterexamples, give a complete model. Show that the model satisfies the premise(s) of the sequent in question, but does not satisfy the respective conclusion.

7.1.1 Proof Rules for Universal Quantification

1. [Lecture] $\forall x\ \big(P(x) \to Q(x)\big), \forall x\ P(x)\ \vdash\ \forall x\ Q(x)$.
2. [Lecture] $\forall x\ P(x) \land \forall x\ (P(y) \to Q(x)) \quad \vdash \quad Q(z)$
3. [Lecture] $\forall x\ P(x) \lor \forall x\ Q(x) \quad \vdash \quad \forall y\ (P(y) \lor Q(y))$

7.1.2 Proof Rules for Existential Quantification

4. [Lecture] $\forall x\ (P(x) \to Q(y)), \forall y\ (P(y) \land R(x)) \quad \vdash \quad \exists x\ Q(x))$
5. [Lecture] $\forall a \forall b\ (P(a) \land Q(b)) \quad \vdash \quad \forall a \exists b\ (P(a) \lor Q(b))$
6. [Lecture] Explain the $\exists$-elimination rule ($\exists$e). Why does this rule require a box and what does it mean that x_0 is *fresh*?
7. [Lecture] $\exists x\ \neg P(x), \forall x\ \neg Q(x) \quad \vdash \quad \exists x\ (\neg P(x) \land \neg Q(x))$
8. [Lecture] Consider the following natural deduction proof for the sequent

$$\forall x\ (P(x) \to Q(x)), \quad \exists x\ P(x) \quad \vdash \quad \forall x Q(x).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.

1.	$\forall x\ (P(x) \to Q(x))$	prem.
2.	$\exists x\ P(x)$	prem.
3.	x_0 (box begins)	
4.	$P(x_0)$ (inner box begins)	ass.
5.	$P(x_0) \to Q(x_0)$	$\forall$e 1
6.	$Q(x_0)$ (inner box ends)	$\to$ e, 4,5
7.	$\forall x\ Q(x)$ (box ends)	$\forall$i 4-6
8.	$\forall x\ Q(x)$	$\exists$e 2,3-7

9. [Lecture] $\exists x\ (P(x) \to Q(y)), \quad \forall x\ P(x) \quad \vdash \quad Q(y)$

7.1.3 Quantifier Equivalences

10. [Lecture] $\forall x\ \neg(P(x) \land Q(x)) \quad \vdash \quad \neg\exists x\ (P(x) \land Q(x))$
11. [Lecture] $\neg\exists x\ (P(x) \land Q(x)) \quad \vdash \quad \forall x\ \neg(P(x) \land Q(x))$

7.1.4 Counterexamples

12. [Lecture] $\exists x\ \neg P(x), \exists x\ \neg Q(x) \quad \vdash \quad \exists x\ (\neg P(x) \land \neg Q(x))$

13. [Lecture] $\exists x\ (P(x) \to Q(y)), \quad \exists x\ P(x) \quad \vdash \quad Q(y)$

7.2 Practicals

For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

For proofs, clearly indicate which rule, and what assumptions/premises/
intermediate results you are using in each step. Also clearly indicate the scope of any boxes you use.

For counterexamples, give a complete model. Show that the model satisfies the premise(s) of the sequent in question, but does not satisfy the respective conclusion.

1. [Practicals] **[2 Point]**
   (a) $(\forall x\ (\neg A(x)) \lor (\exists x\ (B(x)) \quad \vdash \quad \forall x\ (\neg A(x) \lor B(x))$
   (b) $(\forall x\ (\neg A(x)) \lor (\exists x\ (B(x)) \quad \vdash \quad \exists x\ (\neg A(x) \lor B(x))$

2. [Practicals] **[2 Point]** $\exists x P(x) \lor \exists x Q(x) \ \vdash \ \exists x (P(x) \lor Q(x))$

3. [Practicals] **[3 Point]** $\exists b\ (a \to B(b)) \quad \vdash \quad a \to \exists b\ B(b)$

4. [Practicals] **[3 Point]** $\exists x\ (S(x) \to T(x)), \neg T(z) \land \neg T(y) \quad \vdash \quad \neg S(y)$

5. [Practicals] **[3 Point]** $\forall r\ U(r) \land \forall r\ (S(r) \to T(r)) \quad \vdash \quad \exists r\ \neg T(x) \to \exists r (\neg S(r) \land U(r))$

6. [Practicals] **[3.5 Point]** $\exists a\ (P(a) \lor Q(a)), \quad \exists a\ P(a) \to R(c), \quad \exists b\ Q(b) \to R(c) \quad \vdash R(c)$

7. [Practicals] **[3.5 Point]** $\exists x\ P(x) \to \exists x\ Q(x) \quad \vdash \quad \exists x\ (P(x) \to Q(x))$

7.3 Self-Assessment

For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

For proofs, clearly indicate which rule, and what assumptions/premises/
intermediate results you are using in each step. Also clearly indicate the scope of any boxes you use.

For counterexamples, give a complete model. Show that the model satisfies the premise(s) of the sequent in question, but does not satisfy the respective conclusion.

7.3.1 Proof Rules for Universal Quantification

14. [Self-Assessment] Explain the $\forall$-introduction rule and the $\forall$-elimination rule. Explain why one rule needs a box while the other one does not. What does it mean that x_0 needs to be fresh?

15. [Self-Assessment] $\forall x\ (P(x) \land Q(x)) \quad \vdash \quad \forall x\ ((Q(x) \lor R(x)) \land (R(x) \lor P(x)))$

16. [Self-Assessment] $\forall x\ (P(x) \lor Q(x)), \quad \forall x\ (\neg P(x)) \quad \vdash \quad \forall x\ (Q(x))$

7.3.2 Proof Rules for Existential Quantification

17. [Self-Assessment] $\exists x\ (Q(x) \to R(x)),\quad \exists x\ (P(x) \land Q(x)) \quad \vdash \quad \exists x\ (P(x) \land R(x))$
18. [Self-Assessment] $\forall x\ (Q(x) \to R(x)),\quad \exists x\ (P(x) \land Q(x)) \quad \vdash \quad \exists x\ (P(x) \land R(x))$

7.3.3 Quantifier Equivalences

19. [Self-Assessment] $\neg\exists x\forall y\ (P(x) \land Q(y)) \quad \vdash \quad \forall x\exists y\ \neg(P(x) \land Q(y))$
20. [Self-Assessment] $\forall x\exists y\ \neg(P(x) \land Q(y)) \quad \vdash \quad \neg\exists x\forall y\ (P(x) \land Q(y))$

7.3.4 Counterexamples

21. [Self-Assessment] $\neg\exists x\ \neg P(x) \quad \vdash \quad \forall x\ \neg P(x)$

7.3.5 Mixed Examples

22. [Self-Assessment] $P(x) \lor Q(y),\quad P(x) \to R(z),\quad Q(y) \to R(z) \quad \vdash \quad R(z)$
23. [Self-Assessment] $\exists y\forall x\ (P(x,y)) \quad \vdash \quad \forall x\exists y\ (P(x,y))$
24. [Self-Assessment] $\exists a\forall b\ (S(b,a) \land T(b,a)) \quad \vdash \quad \forall b\forall a\ (S(b,a) \land T(b,a))$
25. [Self-Assessment] $P(y) \to \forall x Q(x),\ \exists x \neg Q(x)\ \vdash\ \exists x \neg P(x)$
26. [Self-Assessment] Consider the following natural deduction proof for the sequent

$$\exists x\ \neg P(x) \quad \vdash \quad \neg\forall x\ P(x).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.

1.	$\exists x\ \neg P(x)$	prem.
2.	$\forall x\ P(x)$	ass.
3.	$P(x_0)$	$\forall$e 2
4.	$\exists x\ P(x)$	$\exists$i 3
5.	$\bot$	$\neg$e 1,4
6.	$\neg\forall x\ P(x)$	$\neg$e 2-5

27. [Self-Assessment] Consider the following natural deduction proof for the sequent

$$\exists x\ P(x) \lor \exists x\ Q(x) \quad \vdash \quad \exists x\ (P(x) \lor Q(x)).$$

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.

1.		$\exists x\, P(x) \vee \exists x\, Q(x)$	prem.
2.		$\exists x\, P(x)$	ass.
3.	x_0	$P(x_0)$	ass.
4.		$P(x_0) \vee Q(x_0)$	$\vee$i$_1$ 3
5.		$\exists x\, (P(x) \vee Q(x))$	$\exists$e 2,3-4
6.		$\exists x\, Q(x)$	ass.
7.	x_0	$Q(x_0)$	ass.
8.		$P(x_0) \vee Q(x_0)$	$\vee$i$_2$ 7
9.		$\exists x\, (P(x) \vee Q(x))$	$\exists$e 6,7-8
10.		$\exists x\, (P(x) \vee Q(x))$	$\vee$e 1,2-5,6-9

28. [Self-Assessment] $\forall x \exists y\, (P(x) \to Q(y)), P(s) \quad \vdash \quad \exists x \forall y\, (\neg P(x) \vee Q(y))$