# Logic and Computability SS22
# Assignment 5

Practical Session: June 02, 2022

1. [Practicals] **[3 Points]** Given the formula:

   $\varphi_{EUF} := f(x) = y \wedge x = g(x) \vee x \neq f(x) \wedge g(x) = f(g(x)) \vee y \neq g(x) \wedge x = f(y) \wedge g(y) = f(g(x))$

   Apply the *Ackermann* reduction algorithm to compute an equisatisfiable formula in $\mathcal{T}_E$.
   Solution:

$$\begin{aligned}\varphi_{FC} \quad &:= \quad (x = g_x \rightarrow f_x = f_{g_x}) \wedge \\ &\quad\quad (x = y \rightarrow f_x = f_y) \wedge \\ &\quad\quad (g_x = y \rightarrow f_{g_x} = f_y) \wedge \\ &\quad\quad (x = y \rightarrow g_x = g_y)\end{aligned}$$

$$\hat{\varphi}_{EUF} \quad := \quad f_x = y \wedge x = g_x \vee x \neq f_x \wedge g_x = f_{g_x} \vee y \neq g_x \wedge x = f_y \wedge g_y = f_{g_x}$$

$$\varphi_E \quad := \hat{\varphi}_{EUF} \wedge \varphi_{FC}$$

2. [Practicals] **[3 Points]** Given the formula:

$$\varphi_{EUF} \quad := \quad x = f(x,y) \land x \neq y \leftrightarrow z = f(x,y) \lor f(y,z) \neq z \land y \neq f(x,y) \lor y = f(x,z)$$

Apply the *Ackermann* reduction algorithm to compute an equisatisfiable formula in $\mathcal{T}_E$.
Solution:

$$\begin{aligned} \varphi_{FC} \quad := \quad & (x = y \land y = z \rightarrow f_{xy} = f_{yz}) \land \\ & (x = y \land z = z \rightarrow f_{yz} = f_{xz}) \land \\ & (x = x \land y = z \rightarrow f_{xy} = f_{xz}) \end{aligned}$$

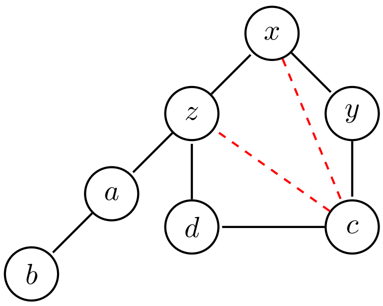
$$\hat{\varphi}_{EUF} \quad := \quad x = f_{xy} \land x \neq y \leftrightarrow z = f_{xy} \lor f_{yz} \neq z \land y \neq f_{xy} \lor y = f_{xz}$$

$$\varphi_E \quad := \hat{\varphi}_{EUF} \land \varphi_{FC}$$

3. [Practicals] **[3 Points]** Perform graph-based reduction to translate the following formula in $\mathcal{T}_E$ into an equisatisfiable formula in propositional logic.

$$\varphi_E := x \neq y \land y = c \lor c = d \to \neg(d \neq z \lor z = a) \land \neg(a = b \land x \neq z).$$

Solution:




$$\begin{aligned}
\varphi_{TC} = \;& ((e_{x=y} \land e_{y=c}) \to e_{x=c}) \land \\
& ((e_{x=y} \land e_{x=c}) \to e_{y=c}) \land \\
& ((e_{x=c} \land e_{y=c}) \to e_{x=y}) \land \\
& ((e_{c=d} \land e_{d=z}) \to e_{c=z}) \land \\
& ((e_{c=d} \land e_{c=z}) \to e_{d=z}) \land \\
& ((e_{c=z} \land e_{d=z}) \to e_{c=d}) \land \\
& ((e_{x=c} \land e_{c=z}) \to e_{x=z}) \land \\
& ((e_{x=c} \land e_{x=z}) \to e_{c=z}) \land \\
& ((e_{x=z} \land e_{c=z}) \to e_{x=c})
\end{aligned}$$

$$\begin{aligned}
\hat{\varphi}_E =& (\neg e_{x=y} \land e_{y=c}) \lor \\
& (e_{c=d} \to (\neg(\neg e_{d=z} \lor e_{z=a}) \land \neg(e_{a=b} \land \neg(e_{x=z}))))
\end{aligned}$$

4. [Practicals] **[5 Points]** Consider the following formula in $\mathcal{T}_{EUF}$:

$$\varphi_{EUF} \quad := (y = z \vee f(x) = f(y)) \rightarrow (x = z \vee f(x) = x \wedge f(x) = y)$$

Use Ackermann's reduction to compute an equisatisfiable formula in $\mathcal{T}_E$.

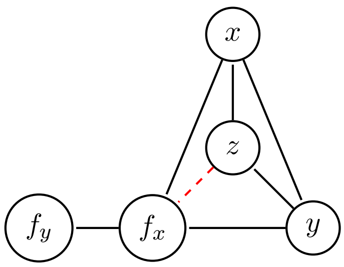
Then perform the graph-based reduction on the outcome of Ackermann's reduction to construct an equisatisfiable propositional formula.

Solution:

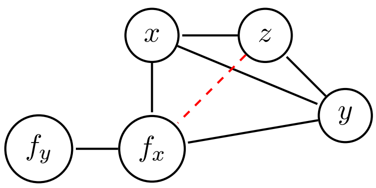
$$\varphi_{FC} = (x = y \rightarrow f_x = f_y)$$
$$\hat{\varphi}_{EUF} = (y = z \vee f_x = f_y) \rightarrow (x = z \vee f_x = x \wedge f_x = y)$$

$$\begin{aligned} \varphi_E = {} & (x = y \rightarrow f_x = f_y) \wedge \\ & (y = z \vee f_x = f_y) \rightarrow (x = z \vee f_x = x \wedge f_x = y) \end{aligned}$$

Version 1:




Version 2:




$$\begin{aligned} \varphi_{TC} = {} & ((e_{x=y} \wedge e_{y=z}) \rightarrow e_{x=z}) \wedge \\ & ((e_{x=y} \wedge e_{x=z}) \rightarrow e_{y=z}) \wedge \\ & ((e_{x=z} \wedge e_{y=z}) \rightarrow e_{x=y}) \wedge \\ & ((e_{x=y} \wedge e_{y=f_x}) \rightarrow e_{x=f_x}) \wedge \\ & ((e_{x=y} \wedge e_{x=f_x}) \rightarrow e_{y=f_x}) \wedge \\ & ((e_{x=f_x} \wedge e_{y=f_x}) \rightarrow e_{x=y}) \wedge \\ & ((e_{y=z} \wedge e_{z=f_x}) \rightarrow e_{y=f_x}) \wedge \\ & ((e_{y=z} \wedge e_{y=f_x}) \rightarrow e_{z=f_x}) \wedge \\ & ((e_{y=f_x} \wedge e_{z=f_x}) \rightarrow e_{y=z}) \end{aligned}$$

$$\begin{aligned} \hat{\varphi}_E = {} & (e_{x=y} \rightarrow e_{f_x=f_y}) \wedge \\ & (e_{y=z} \vee e_{f_x=f_y}) \rightarrow \\ & (e_{x=z} \vee e_{f_x=x} \wedge e_{f_x=y}) \end{aligned}$$

$$\varphi_{prop} = \varphi_{TC} \wedge \hat{\varphi}_E$$

5. [Practicals] **[3 Points]** Use the Congruence-Closure algorithm to check if the following assignment for the equalities is satisfiable.

$$\varphi_{EUF} \quad := \quad f(b) = a \wedge e = b \wedge c = f(c) \wedge d \neq f(e) \wedge f(a) = f(d) \wedge a \neq f(c) \wedge d = f(a)$$

Solution:

$$\{f(b), a\}, \{e, b\}, \{c, f(c)\}, \{f(e)\}, \{\underline{f(a)}, f(d)\}, \{d, \underline{f(a)}\}$$
$$\{\underline{f(b)}, a\}, \{\underline{e, b}\}, \{c, f(c)\}, \{\underline{f(e)}\}, \{f(a), f(d), d\}$$
$$\{f(b), a, f(e)\}, \{e, b\}, \{c, f(c)\}, \{f(a), f(d), d\}$$

Checking the disequalities $d \neq f(e)$ and $a \neq f(c)$ leads to the result that the assignment is SAT, since neither $d$ and $f(e)$ nor $a$ and $f(c)$ are in the same congruence class.

6. [Practicals] **[3 Points]** Use the Congruence-Closure algorithm to check if the following assignment for the equalities is satisfiable.

$$\begin{aligned}\varphi_{EUF} \quad := \quad & f(o) = k \land l \neq f(m) \land n \neq l \land f(k) = m \land f(o) = f(k) \land o \neq k \land \\ & l \neq f(n) \land f(m) \neq k \land m \neq f(m) \land o = n \land f(m) = o\end{aligned}$$

Solution:

$$\{k, \underline{f(o)}\}, \{l\}, \{m, f(k)\}, \{f(k), \underline{f(o)}\}, \{f(n)\}, \{n, o\}, \{o, f(m)\}$$
$$\{k, f(k), f(o)\}, \{l\}, \{m, f(k)\}, \{f(n)\}, \{n, \underline{o}\}, \{\underline{o}, f(m)\}$$
$$\{k, \underline{f(k)}, f(o)\}, \{l\}, \{m, \underline{f(k)}\}, \{f(n)\}, \{n, o, f(m)\}$$
$$\{k, m, f(k), \underline{f(o)}\}, \{l\}, \{\underline{f(n)}\}, \{\underline{n, o}, f(m)\}$$
$$\{\underline{k, m}, \underline{f(k)}, f(n), f(o)\}, \{l\}, \{n, o, \underline{f(m)}\}$$
$$\{k, m, n, o, f(k), f(m), f(n), f(o)\}, \{l\}$$

Checking the disequalities $o \neq k$, $f(m) \neq k$, $m \neq f(m)$ leads to the result that the assignment is UNSAT, since $o$ and $k$, $f(m)$ and $k$, $m$ and $f(m)$ are in the same congruence class.