Logic and Computability SS22
Assignment 4

Practical Session: June 02, 2022

For each of the following sequents, either provide a natural deduction proof, or a counterexample that proves the sequent invalid.

For proofs, clearly indicate which rule, and what assumptions/premises/
intermediate results you are using in each step. Also clearly indicate the scope of any boxes you use.

For counterexamples, give a complete model. Show that the model satisfies the premise(s) of the sequent in question, but does not satisfy the respective conclusion.

1. [Practicals] **[2 Point]**

   (a) $(\forall x\ (\neg A(x)) \vee (\exists x\ (B(x)) \quad \vdash \quad \forall x\ (\neg A(x) \vee B(x))$

   (b) $(\forall x\ (\neg A(x)) \vee (\exists x\ (B(x)) \quad \vdash \quad \exists x\ (\neg A(x) \vee B(x))$

Solution:

(a) This sequent is not provable.
Model $\mathcal{M}$:

$$\mathcal{A} = \{a, b\}$$
$$A^{\mathcal{M}} = \{a, b\}$$
$$B^{\mathcal{M}} = \{a\}$$

$\mathcal{M} \models \forall x\ (\neg A(x)) \vee \exists x\ (B(x))$
$\mathcal{M} \not\models \forall x\ (\neg A(x) \vee B(x))$

(b) This sequent is provable.

1.		$\forall x\ \neg A(x) \vee \exists x\ B(x)$	prem.
2.		$\forall x\ \neg A(x)$	ass.
3.		$\neg A(x_0)$	$\forall$e 2
4.		$\neg A(x_0) \vee B(x_0)$	$\vee$i 3
5.		$\exists x\ (\neg A(x) \vee B(x))$	$\exists$i4
6.		$\exists x\ B(x)$	ass.
7.	x_0	$B(x_0)$	ass.
8.		$\neg A(x_0) \vee B(x_0)$	$\vee$i 7
9.		$\exists x(\neg A(x) \vee B(x))$	$\exists$i 8
10.		$\exists x\ (\neg A(x) \vee B(x))$	$\exists$e 6, 7-9
11.		$\exists x\ (\neg A(x) \vee B(x))$	$\vee$e 1, 2-5, 6-10

2. [Practicals] **[2 Point]** $\exists x P(x) \lor \exists x Q(x) \;\vdash\; \exists x(P(x) \lor Q(x))$
Solution:

1.		$\exists x\ P(x) \lor \exists x\ Q(x)$	prem.
2.		$\exists x\ P(x)$	ass.
3.	x_0	$P(x_0)$	ass.
4.		$P(x_0) \lor Q(x_0)$	$\lor$i 3
5.		$\exists x(P(x) \lor Q(x))$	$\exists$i 4
6.		$\exists x\ (P(x) \lor Q(x))$	$\exists e2, 3 - 5$
7.		$\exists x\ Q(x)$	ass.
8.	x_0	$Q(x_0)$	ass.
9.		$P(x_0) \lor Q(x_0)$	$\lor$i 7
10.		$\exists x(P(x) \lor Q(x))$	$\exists$i 8
11.		$\exists x\ (P(x) \lor Q(x))$	$\exists$e 6, 7-9
12.		$\exists x\ (P(x) \lor Q(x))$	$\lor$e 1, 2-5, 6-10

3. [Practicals] **[3 Point]** $\exists b\ (A \to B(b)) \quad \vdash \quad A \to \exists b\ B(b)$
Solution:

1.		$\exists b\ (A \to B(b))$	prem.
2.	t	$A \to B(t)$	ass.
3.		A	ass.
4.		$B(t)$	$\to$ e 3,2
5.		$\exists b\ B(b)$	$\exists$i 4
6.		$A \to \exists b\ B(b)$	$\to$ i 3-5
7.		$A \to \exists b\ B(b)$	$\exists$e 1, 2-6

4. [Practicals] **[4 Point]** $\exists x\ (S(x) \to T(x)), \neg T(z) \land \neg T(y) \quad \vdash \quad \neg S(y)$
Solution:

This sequent is not provable.
Model $\mathcal{M}$:

$$\mathcal{A} = \{a, y, z\}$$
$$S^{\mathcal{M}} = \{a, y\}$$
$$T^{\mathcal{M}} = \{a\}$$

5. [Practicals] **[4 Point]** $\forall r\ U(r) \wedge \forall r\ (S(r) \rightarrow T(r)) \quad \vdash \quad \exists r\ \neg T(r) \rightarrow \exists r(\neg S(r) \wedge U(r))$
Solution:

1.		$\forall r\ U(r) \wedge \forall r\ (S(r) \rightarrow T(r))$	prem
2.		$\exists r \neg T(r)$	ass
3.	r_0	$\neg T(r_0)$	ass
4.		$\forall r\ (S(r) \rightarrow T(r))$	$\wedge$e 1
5.		$S(r_0) \rightarrow T(r_0)$	$\forall$e 5
6.		$\neg S(r_0)$	MT 6, 4
7.		$\forall r\ U(r)$	$\wedge$e 1
8.		$U(r_0)$	$\forall$e 8
9.		$\neg S(r_0) \wedge U(r_0)$	$\wedge$i 7, 9
10.		$\exists r\ (\neg S(r) \wedge U(r))$	$\exists$i 10
11.		$\exists r\ (\neg S(r) \wedge U(r))$	$\exists$e 2, 3-11
12.		$\exists r\ \neg T(r) \rightarrow \exists r(\neg S(r) \wedge U(r))$	$\rightarrow$ i 2-12

6. [Practicals] **[5 Point]** $\exists a\ (P(a) \vee Q(a)),\quad \exists a\ P(a) \rightarrow R(c),\quad \exists b\ Q(b) \rightarrow R(c) \quad \vdash R(c)$
Solution:

1.		$\exists a\ (P(a) \vee Q(a))$	prem.
2.		$\exists a\ P(a) \rightarrow R(c)$	prem.
3.		$\exists b\ Q(b)) \rightarrow R(c)$	prem.
4.	t	$P(t) \vee Q(t)$	ass
5.		$P(t)$	ass
6.		$\exists a\ P(a)$	$\exists$i 6
7.		$R(c)$	$\rightarrow$ e 7, 2
8.		$Q(t)$	ass
9.		$\exists b\ Q(b)$	$\exists$i 9
10.		$R(c)$	$\rightarrow$ e 10, 3
11.		$R(c)$	$\vee$e 5, 6-8, 9-11
12.		$R(c)$	$\exists$e 1, 4-12

7. [Practicals] **[5 Point]** $\exists x\ P(x) \to \exists x\ Q(x) \quad \vdash \quad \exists x\ (P(x) \to Q(x))$

Solution:

1.		$\exists x\ P(x) \to \exists x\ Q(x)$	prem.
2.		$\exists x\ P(x) \lor \neg\exists x\ P(x)$	LEM
3.		$\exists x\ P(x)$	ass.
4.		$\exists x\ Q(x)$	$\to$ e 3,1
5.	x_0	$Q(x_0)$	ass.
6.		$P(x_0)$	ass.
7.		$Q(x_0)$	copy 5
8.		$P(x_0) \to Q(x_0)$	$\to$ i 6-7
9.		$\exists x\ (P(x) \to Q(x))$	$\exists$i 8
10.		$\exists x\ (P(x) \to Q(x))$	$\exists$e 4, 5-9
11.		$\neg\exists x\ P(x)$	ass.
12.		$P(x_0)$	ass.
13.		$\exists x\ P(x)$	$\exists$i 12
14.		$\bot$	$\neg$e 11,14
15.		$\neg P(x_0)$	$\neg$i 12-14
16.		$P(x_0)$	ass.
17.		$\bot$	$\neg$e 15,16
18.		$Q(x_0)$	$\bot$e 17
19.		$P(x_0) \to Q(x_0)$	$\to$ i16-18
20.		$\exists x\ (P(x) \to Q(x))$	$\exists$i 19
21.		$\exists x\ (P(x) \to Q(x))$	$\neg$e 1, 2-6