

Logic and Computability SS22

Assignment Sheet 1 - Solutions

Practical Session: 01. 04. 2022

For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

For proofs, clearly indicate which rule, and what assumptions/premises/intermediate results you are using in each step. Also clearly indicate the scope of any boxes you use.

For counterexamples, give a complete model. Show that the model satisfies the premise(s) of the sequent in question, but does not satisfy the respective conclusion. For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

1. **[Practicals] [2 Points]**

- (a) If I am ill, I go to the doctor.
I am ill.
Therefore, I go to the doctor.
- (b) If I am ill, I go to the doctor.
I go to the doctor.
Therefore, I am ill.
- (c) (Solve without using the Modus Tollens)
If I am ill, I go to the doctor.
I did not go to the doctor.
Therefore, I am not ill.

Solution:

Translation:

p : I am ill.

q : I go to the doctor.

- (a) If I am ill, I go to the doctor. $p \rightarrow q$
I am ill. p
Therefore, I go to the doctor. $\vdash q$

Sequent: $p \rightarrow q, p \vdash q$
This sequent is provable.

1. $p \rightarrow q$ prem.
2. p prem.
3. q \rightarrow e 2,1

- (b) If I am ill, I go to the doctor. $p \rightarrow q$
I go to the doctor. q
Therefore, I am ill. $\vdash p$

Sequent: $p \rightarrow q, q \vdash p$
This sequent is not provable.

$\mathcal{M} : p = F, q = T$
 $\mathcal{M} \models p \rightarrow q, q$
 $\mathcal{M} \not\models p$

- (c) (Solve without using the Modus Tollens)
If I am ill, I go to the doctor. $p \rightarrow q$
I did not go to the doctor. $\neg q$
Therefore, I am not ill. $\neg p$

Sequent: $p \rightarrow q, \neg q \vdash \neg p$
This sequent is provable.

1. $p \rightarrow q$ prem.
2. $\neg q$ prem.
3. p ass.
4. q \rightarrow e 3,1
5. \perp \neg e 4,2
6. $\neg p$ \neg i 3-5

(First version)

- If I am ill, I go to the doctor. $p \rightarrow q$
I am not ill. $\neg p$
Therefore, I don't go to the doctor. $\neg q$

Sequent: $p \rightarrow q, \neg p \vdash \neg q$
This sequent is not provable.

$\mathcal{M} : p = F, q = T$
 $\mathcal{M} \models p \rightarrow q, \neg p$
 $\mathcal{M} \not\models \neg q$

2. [Practicals] [2 Points]

- (a) $(p \wedge q) \wedge \neg r \vdash q \vee r$
 (b) $(p \vee q) \wedge \neg r \vdash q \wedge r$

Solution:

(a) This sequent is provable.

1.	$(p \wedge q) \wedge \neg r$	prem.
2.	$p \wedge q$	$\wedge e1$ 1
3.	q	$\wedge e2$ 2
4.	$q \vee r$	$\vee i$ 3

(b) This sequent is not provable.
 $\mathcal{M} : p = T, q = T, r = F$
 $\mathcal{M} \models (p \vee q) \wedge \neg r$
 $\mathcal{M} \not\models q \wedge r$

3. [Practicals] [2 Points]

- (a) $\vdash (p \rightarrow q) \rightarrow p$
 (b) $\vdash p \rightarrow (q \rightarrow p)$

Solution:

(a) This sequent is not provable.
 $\mathcal{M} : p = F, q = F$
 $\mathcal{M} \not\models (p \rightarrow q) \rightarrow p$

(b) This sequent is provable.

1.	p	ass.
2.	q	ass.
3.	p	copy1
4.	$q \rightarrow p$	$\rightarrow i$ 2-3
5.	$p \rightarrow (q \rightarrow p)$	$\rightarrow i$ 1-4

4. [Practicals] [2 Points] $\neg(a \wedge b) \vee \neg c \vdash \neg(a \wedge b) \rightarrow c \vee a$

Solution:

This sequent is not provable.
 $\mathcal{M} : a = F, b = F, c = F$
 $\mathcal{M} \models \neg(a \wedge b) \vee \neg c$
 $\mathcal{M} \not\models \neg(a \wedge b) \rightarrow c \vee a$

5. [Practicals] [2 Points] $p \wedge q \vee r \vdash (p \vee r) \wedge (q \vee r)$

Solution:

This sequent is provable.

1.	$(p \wedge q) \vee r$	prem.
2.	$p \wedge q$	ass.
3.	p	$\wedge e1$ 2
4.	$p \vee r$	$\vee i1$ 3
5.	q	$\wedge e2$ 2
6.	$q \vee r$	$\vee i1$ 5
7.	$(p \vee r) \wedge (q \vee r)$	$\wedge i$ 4,6
8.	r	ass.
9.	$p \vee r$	$\vee i2$ 8
10.	$q \vee r$	$\vee i2$ 8
11.	$(p \vee r) \wedge (q \vee r)$	$\wedge i$ 9,10
12.	$(p \vee r) \wedge (q \vee r)$	$\vee e$ 1, 2-7, 8-11

6. [Practicals] [2 Points] $\neg\neg x \rightarrow \neg y \wedge z \vdash z \rightarrow \neg x \wedge \neg\neg y$

Solution:

This sequent is not provable.

$\mathcal{M} : x = F, y = F, z = T$

$\mathcal{M} \models \neg\neg x \rightarrow \neg y \wedge z$

$\mathcal{M} \not\models z \rightarrow \neg x \wedge \neg\neg y$

7. [Practicals] [2 Points] $\vdash \neg(p \wedge q) \vee p$

Solution:

This sequent is provable.

1.	$p \vee \neg p$	LEM
2.	p	ass.
3.	$\neg(p \wedge q) \vee p$	$\vee i2$ 2
4.	$\neg p$	ass.
5.	$p \wedge q$	ass.
6.	p	$\wedge e1$ 5
7.	\perp	$\neg e$ 6,4
8.	$\neg(p \wedge q)$	$\neg i$ 5-7
9.	$\neg(p \wedge q) \vee p$	$\vee i1$ 8
10.	$\neg(p \wedge q) \vee p$	$\vee e$ 1, 2-3, 4-9

8. [Practicals] [2 Points] $\neg(a \vee b) \vdash \neg a \wedge \neg b$

Solution:

This sequent is provable.

- | | | |
|-----|------------------------|-----------------------|
| 1. | $\neg(a \vee b)$ | prem. |
| 2. | $a \vee \neg a$ | LEM |
| 3. | a | ass. |
| 4. | $a \vee b$ | $\vee i1$ 3 |
| 5. | \perp | $\neg e$ 4,1 |
| 6. | $\neg a \wedge \neg b$ | $\wedge e$ 5 |
| 7. | $\neg a$ | ass. |
| 8. | b | ass. |
| 9. | $a \vee b$ | $\vee i2$ 8 |
| 10. | \perp | $\neg e$ 9,1 |
| 11. | $\neg b$ | $\neg i$ 8-10 |
| 12. | $\neg a \wedge \neg b$ | $\wedge i$ 7, 11 |
| 13. | $\neg a \wedge \neg b$ | $\vee e$ 2, 3-6, 7-12 |

9. [Practicals] [2 Points] $(s \vee \neg u) \rightarrow t \vdash (\neg s \wedge u) \vee t$

Solution:

This sequent is provable.

- | | | |
|-----|---------------------------------|-------------------------|
| 1. | $(s \vee \neg u) \rightarrow t$ | prem. |
| 2. | $s \vee \neg s$ | LEM |
| 3. | s | ass. |
| 4. | $s \vee \neg u$ | $\vee i1$ 3 |
| 5. | t | $\rightarrow e$ 4, 1 |
| 6. | $(\neg s \wedge u) \vee t$ | $\vee i2$ 5 |
| 7. | $\neg s$ | ass. |
| 8. | $u \vee \neg u$ | LEM |
| 9. | u | ass. |
| 10. | $\neg s \wedge u$ | $\wedge i$ 7,9 |
| 11. | $(\neg s \wedge u) \vee t$ | $\vee i1$ 10 |
| 12. | $\neg u$ | ass. |
| 13. | $s \vee \neg u$ | $\vee i2$ 12 |
| 14. | t | $\rightarrow e$ 13 1 |
| 15. | $(\neg s \wedge u) \vee t$ | $\vee i2$ 14 |
| 16. | $(\neg s \wedge u) \vee t$ | $\vee e$ 8, 9-11, 12-15 |
| 17. | $(\neg s \wedge u) \vee t$ | $\vee e$ 2, 3-6, 7-16 |

10. [Practicals] [2 Points] $\neg\neg k \rightarrow (l \vee m), \neg\neg\neg l \rightarrow m \vdash \neg k \vee (l \vee \neg\neg m)$

Solution:

This sequent is provable.

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|-----|---------------------------------------|-----------------------|
| 1. | $\neg\neg k \rightarrow (l \wedge m)$ | prem. |
| 2. | $\neg\neg\neg l \rightarrow m$ | prem. |
| 3. | $m \vee \neg m$ | LEM |
| 4. | m | ass. |
| 5. | $\neg\neg m$ | $\neg\neg$ i 4 |
| 6. | $l \vee \neg\neg m$ | \vee i2 5 |
| 7. | $\neg k \vee (l \vee \neg\neg m)$ | \vee i2 6 |
| 8. | $\neg m$ | ass. |
| 9. | $\neg\neg\neg\neg l$ | MT 2, 8 |
| 10. | $\neg\neg l$ | $\neg\neg$ e 9 |
| 11. | l | $\neg\neg$ e 10 |
| 12. | $l \vee \neg\neg m$ | \vee i1 11 |
| 13. | $\neg k \vee (l \vee \neg\neg m)$ | \vee i2 12 |
| 14. | $\neg k \vee (l \vee \neg\neg m)$ | \vee e 3, 4-7, 8-13 |