Training for Test 2

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For each of the following sequents, either provide a natural deduction proof, or a counterexample that proves the sequent invalid.

- $p \vee q \quad \vdash \quad p \rightarrow q$
- $\neg(p \wedge q) \quad \vdash \quad p \rightarrow \neg q$

p	q	p∨q	p→q
0	0	0	1
0	1	1	1
1	0	1	0
1	1	1	1

$\mathcal{M}$: $p = \top \quad q = \bot$

$\mathcal{M} \models p \vee q$

$\mathcal{M} \not\models p \rightarrow q$

Warm up

1.	$\neg(p \wedge q)$	prem
2.	p	ass
3.	q	ass
4.	$p \wedge q$	$\wedge$i 2,3
5.	$\bot$	$\neg$e 1,4
6.	$\neg q$	$\neg$i 3–5
7	$p \rightarrow \neg q$	$\rightarrow$i 2–6

Test 2 for the Practicals - Predicate Logic

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[**Example 2:**] For each of the following sequents, either provide a natural deduction proof, or a counterexample that proves the sequent invalid.

1. $\exists x(P(x) \rightarrow Q(y)), \exists x P(x) \quad \vdash \quad Q(y)$
2. $\exists x(P(x) \rightarrow Q(y)), \forall x P(x) \quad \vdash \quad Q(y)$

For each of the following sequents, either provide a natural deduction proof, or a counterexample that proves the sequent invalid.

(a) $\exists x(T(x) \wedge S(x)), \quad \forall x(S(x) \rightarrow R(x)) \quad \vdash \quad \exists x(T(x) \wedge R(x))$

(b) $\exists x(T(x) \wedge S(x)), \quad \exists x(S(x) \rightarrow R(x)) \quad \vdash \quad \exists x(T(x) \wedge R(x))$

For each of the following sequents, either provide a natural deduction proof, or a counterexample that proves the sequent invalid.

(a) $\forall x(P(x) \rightarrow \neg Q(x)) \quad \vdash \quad \forall x(P(x)) \rightarrow \forall x(\neg Q(x))$

(b) $\exists x(P(x) \rightarrow \neg Q(x)) \quad \vdash \quad \exists x(P(x)) \rightarrow \exists x(\neg Q(x))$

Consider the following natural deduction proof for the sequent

$$\exists x \neg[\neg Q(x) \vee P(x)] \vdash \neg\forall x[Q(x) \rightarrow P(x)].$$

The prove is incomplete. Your task is to complete the proof: clearly indicate which rule, and what assumptions/premises/intermediate results is used in each step.

1	$\exists x\ \neg(\neg Q(x) \vee P(x))$	premise
2	$\neg(\neg Q(x_0) \vee P(x_0))$	
3	$\forall x\ (Q(x) \rightarrow P(x))$	
4	$Q(x_0) \rightarrow P(x_0)$	
5	$Q(x_0) \vee \neg Q(x_0)$	
6	$Q(x_0)$	
7	$P(x_0)$	
8	$\neg Q(x_0) \vee P(x_0)$	
9	$\neg Q(x_0)$	
10	$\neg Q(x_0) \vee P(x_0)$	
11	$\neg Q(x_0) \vee P(x_0)$	
12	$\bot$	
13	$\neg\forall x\ (Q(x) \rightarrow P(x))$	
14	$\neg\forall x\ (Q(x) \rightarrow P(x))$	

Is the proof correct? If not, explain the error in the proof and either show how to correctly prove the sequent, or give a counterexample that proves the sequent invalid.

1	$\exists x P(x) \vee \exists x Q(x)$	premise
2	$\exists x P(x)$	assumption,
3	$P(x_0)$	assumption, x_0 fresh
4	$P(x_0) \vee Q(x_0)$	$\vee$i, 3
5	$\exists x(P(x) \vee Q(x))$	$\exists$e, 2, 3–4
6	$\exists x Q(x)$	assumption,
7	$Q(x_0)$	assumption, x_0 fresh
8	$P(x_0) \vee Q(x_0)$	$\vee$i, 7
9	$\exists x(P(x) \vee Q(x))$	$\exists$e, 6, 7–8
10	$\exists x(P(x) \vee Q(x))$	$\vee$e, 1, 2–5, 6–9

Consider the following formula in the conjunctive fragment of $\mathcal{T}_{UE}$. Let $a \neq b$ be a shorthand notation for $\neg(a = b)$.

$$a = b \wedge f(c) = b \wedge c = d \wedge e \neq a \wedge f(a) \neq f(e) \wedge$$
$$f(e) = f(b) \wedge f(d) = f(a) \wedge f(b) \neq f(d) \wedge f(c) = c$$

Use the *Congruence Closure* algorithm to determine whether or not this formula is satisfiable.

SMT-Lazy Encoding

Training for Exam

Consider the propositional formula $\varphi = p \rightarrow (q \rightarrow r)$.

(a) Fill out the truth table for φ (and its subformulas). Write **T** for true and **F** for false.

Propositional Logic

p	q	r	$(q \rightarrow r)$	$\varphi = p \rightarrow (q \rightarrow r)$
F	F	F		
F	F	T		
F	T	F		
F	T	T		
T	F	F		
T	F	T		
T	T	F		
T	T	T		

(b) Is φ satisfiable?

(c) Is φ valid?

(d) Give a formula ψ that is semantically equivalent to φ, but does not use the "$\rightarrow$" connective.

(e) How can you check whether ψ is semantically equivalent to φ?

Prop + Pred Logic

Test 2 for the Practicals - Predicate Logic

[**Example 1:**] Model the following sentences with predicate logic, as detailed as possible. Clearly indicate the intended meaning of all function, predicate, and constant symbols that you use.

Also, model the same sentence in propositional logic, as detailed as possible. Clearly indicate the intended meaning of each propositional variable you use.

a) *"Every even integer greater than 2 is equal to the sum of two prime numbers."*

b) *"Not all birds can fly, but some birds can fly."*

1. Reduced and Ordered **Binary Decision Diagrams** (BDDs).

(a) For each type of node that can occur in a BDD, state how many incoming and outgoing edges this type of node has. Write your answers in the following table. If the number of certain edges is always a fixed number, write that number. If the number of certain edges can be arbitrary, write "A".

Node Type	Incoming Edges	Outgoing Edges
Function Node		
Internal Node		
Terminal Node		

BDD

(b) Use the following BDD to check if the function it represents evaluates to *true* or *false* with the following assignments for the variables. Tic the correct answers.

- ☐ The BDD evaluates to *true* with the assignment $a = \top, b = \bot, c = \top$.
- ☐ The BDD evaluates to *false* with the assignment $a = \top, b = \bot, c = \top$.
- ☐ The BDD evaluates to *true* with the assignment $a = \bot, b = \top, c = \bot$
- ☐ The BDD evaluates to *false* with the assignment $a = \bot, b = \top, c = \bot$

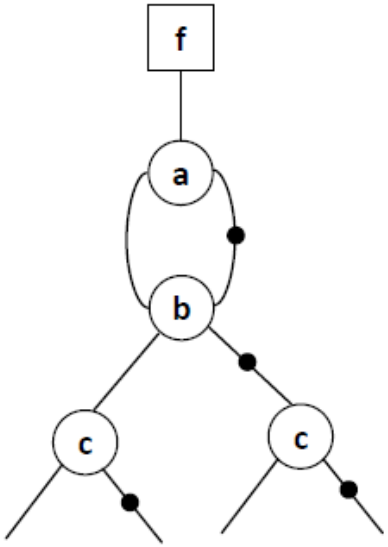

Figure 1: BDD

BDD

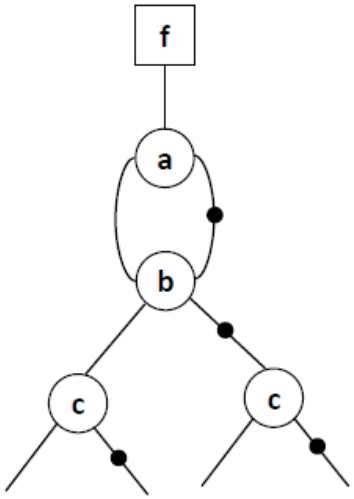

Figure 1: BDD

(c) Tic the correct propositional formula for the function f that is represented by the BDD in Figure 1.

- ☐ $(a \wedge b \wedge c) \vee (\neg a \wedge \neg b \wedge \neg c)$
- ☐ $(a \wedge b \wedge c) \vee (a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c)$
- ☐ $(a \vee b \vee c) \wedge (\neg a \vee \neg b \vee \neg c)$
- ☐ $(a \vee b \vee c) \wedge (a \vee \neg b \vee \neg c) \wedge (\neg a \vee b \vee \neg c) \wedge (\neg a \vee \neg b \vee c)$

(d) Is the BDD in Figure 1 reduced? Tic the correct answer.

- ☐ Yes.
- ☐ No.

Consider the domain $A = \{Spain, France, Italy, Germany\}$ and the two different symbolic encodings for A given below. Which one gives a shorter characteristic function for the set $B = \{France, Germany\}$? Illustrate your answer by giving the characteristic function for B in both encodings.

Symbolic Encoding

Encoding 1		
Element	x	y
Spain	0	0
France	1	0
Italy	0	1
Germany	1	1

Encoding 2		
Element	x	y
Spain	0	0
France	1	0
Italy	1	1
Germany	0	1

SMT

The Graph Based Reduction is used to reduce a formula φ_{in} in _____ to a formula in _____ that is equisatisfiable. Two formulas are equisatisfiable if _____ . In the first step of the algorithm, we create a Non-Polar Equality Graph and in the next step we make it chordal. The graph is chordal, if _____. We introduce fresh propositional variables for each equation to ensure _____. In order to ensure transitivity, the algorithm adds constraints of the form _____ for all _____ in the graph. The resulting equisatisfiable formula consists of two parts and is of the form: $\varphi_{out} := \varphi_{TC} \_\_ \hat{\varphi_{in}}$. The right part of the formula $\hat{\varphi_{in}}$ describes the flattening original formula in which we replace _____ with _____ .

For the following example, the SMT solver Z3 is not able to return a *model*. In general, what does Z3 understand under a model? Why is there no model for this example?

Bonus-SMT-Solver

```
1 (declare-fun x () Bool)
2 (assert (and x (not x)))
3 (check-sat)
4 (get-model)
```