Logic and Computability SS21,
Assignment 4,
Solution

Deadline: 2021-06-03 3:59am

For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.
For proofs, clearly indicate which rule, and what assumptions/premises/ intermediate results you are using in each step. Also clearly indicate the scope of any boxes you use.
For counterexamples, give a complete model. Show that the model satisfies the premise(s) of the sequent in question, but does not satisfy the respective conclusion.

20. **[1 Point]** $\forall x P(x) \land \forall x Q(x) \ \vdash \ \forall y \big(P(y) \lor Q(y)\big)$

1.		$\forall x P(x) \land \forall x Q(x)$	premise
2.		$\forall x P(x)$	$\land e_1$ 1
3.	t fresh	$P(t)$	$\forall e$ 2
4.		$P(t) \lor Q(t)$	$\lor i_1$ 3
5.		$\forall y \big(P(y) \lor Q(y)\big)$	$\forall i$ 3-4

(Lines 3–4 are enclosed in a box.)

21. **[1 Point]** $\exists x P(x) \land \exists x Q(x) \ \vdash \ \exists y \big(P(y) \lor Q(y)\big)$

1.		$\exists x P(x) \land \exists x Q(x)$	premise
2.		$\exists x P(x)$	$\land e_1$ 1
3.	t fresh	$P(t)$	assumption
4.		$P(t) \lor Q(t)$	$\lor i_1$ 3
5.		$\exists y \big(P(y) \lor Q(y)\big)$	$\exists i$ 4
6.		$\exists y \big(P(y) \lor Q(y)\big)$	$\exists e$ 2,3-5

(Lines 3–5 are enclosed in a box.)

22. **[1 Point]** $\exists x\big(\neg P(x) \lor Q(x)\big) \;\vdash\; \forall x\big(P(x) \to Q(x)\big)$

Model M: $A = \{a, b\}$
$P^M = \{a, b\}$
$Q^M = \{a\}$

$M \models \exists x\big(\neg P(x) \lor Q(x)\big)$
$M \not\models \forall x\big(P(x) \to Q(x)\big)$

$\exists x\big(\neg P(x) \lor Q(x)\big) \;\nvdash\; \forall x\big(P(x) \to Q(x)\big)$
M is a counterexample.

23. **[1 Point]** $\forall x\big(\neg P(x) \lor Q(x)\big) \;\vdash\; \exists x\big(P(x) \to Q(x)\big)$

1.	$\forall x\big(\neg P(x) \lor Q(x)\big)$	premise
2.	$\neg P(t) \lor Q(t)$	$\forall$e 1
3.	$P(t)$	assumption
4.	$\neg P(t)$	assumption
5.	$\bot$	$\neg$e 3,4
6.	$Q(t)$	$\bot$e 5
7.	$Q(t)$	assumption
8.	$Q(t)$	$\lor$e 2,4-6,7
9.	$P(t) \to Q(t)$	$\to$i 3-8
10.	$\exists x\big(P(x) \to Q(x)\big)$	$\exists$i 9

24. **[1 Point]** $\forall x P(x) \to Q(x, y), \exists x \exists y\, Q(x, y) \;\vdash\; \exists x\, \neg P(x)$

Model M: $A = \{a, b\}$
$P^M = \top$
$Q^M = \top$

$M \models \forall x\, P(x) \to Q(x, y)$
$M \models \exists x \exists y\, Q(x, y)$
$M \not\models \exists x\, \neg P(x)$

$\forall x P(x) \to Q(x, y), \exists x \exists y\, Q(x, y) \;\nvdash\; \exists x\, \neg P(x)$
M is a counterexample.

25. **[1 Point]** $\forall x \exists y \big(f(x) = y\big) \;\vdash\; \forall y \exists x \big(f(x) = y\big)$

Model M:
$$\begin{aligned} A &= \{0,1\} \\ x &= \{0,1\} \\ y &= \{0,1\} \\ f(x) &= 1 \end{aligned}$$

$M \models \forall x \exists y \big(f(x) = y\big)$
$M \not\models \forall y \exists x \big(f(x) = y\big)$

$\forall x \exists y \big(f(x) = y\big) \;\nvdash\; \forall y \exists x \big(f(x) = y\big)$
M is a counterexample.

26. **[1 Point]** $\exists x \forall y\, \neg P(x,y) \;\vdash\; \forall y \exists x\, \neg\big(P(x,y) \wedge Q(y,x)\big)$

1.		$\exists x \forall y\, \neg P(x,y)$	premise
2.	x_0 fresh	$\forall y\, \neg P(x_0,y)$	assumption
3.	y_0 fresh	$\neg P(x_0,y_0)$	$\forall$e 2
4.		$P(x_0,y_0) \wedge Q(y_0,x_0)$	assumption
5.		$P(x_0,y_0)$	$\wedge$e 4
6.		$\bot$	$\neg$e 3,5
7.		$\neg\big(P(x_0,y_0) \wedge Q(y_0,x_0)\big)$	$\neg$i 4-6
8.		$\exists x\, \neg\big(P(x,y_0) \wedge Q(y_0,x)\big)$	$\exists$i 5
9.		$\forall y \exists x\, \neg\big(P(x,y) \wedge Q(y,x)\big)$	$\forall$i 3-8
10.		$\forall y \exists x\, \neg\big(P(x,y) \wedge Q(y,x)\big)$	$\exists$e 1,2-9

27. **[1 Point]** $\exists x \big(P(x) \wedge \neg Q(y)\big) \;\vdash\; \neg \forall x \big(P(x) \to Q(y)\big)$

1.		$\exists x\big(P(x) \land \neg Q(y)\big)$	premise
2.		$\forall x\big(P(x) \to Q(y)\big)$	assumption
3.	x_0 fresh	$P(x_0) \land \neg Q(y)$	assumption
4.		$P(x_0) \to Q(y)$	$\forall$e 2
5.		$P(x_0)$	$\land$e$_1$ 3
6.		$\neg Q(y)$	$\land$e$_2$ 3
7.		$Q(y)$	$\to$e 4,5
8.		$\bot$	$\neg$e 6,7
9.		$\bot$	$\exists$e 1,3-8
10.		$\neg\forall x\big(P(x) \to Q(y)\big)$	$\neg$i 2-9

28. **[2 Points]** $\exists x\big(f(x) = z\big), \forall y\big(f(y) = z\big) \ \vdash \ \forall g \exists h\big(f(g) = f(h)\big)$

1.		$\exists x\big(f(x) = z\big)$	premise
2.		$\forall y\big(f(y) = z\big)$	premise
3.	s fresh	$f(s) = z$	$\forall$e 2
4.	t fresh	$f(t) = z$	assumption
5.		$f(s) = f(t)$	=e 3,4
6.		$\exists h\big(f(s) = f(h)\big)$	$\exists$i 5
7.		$\exists h\big(f(s) = f(h)\big)$	$\exists$e 1,4-6
8.		$\forall g \exists h\big(f(g) = f(h)\big)$	$\forall$i 3-7

29. **[2 Points - Bonus]** $\neg\big(y > (x + x) \lor x = 2\big) \to \neg(y = y)$
$\vdash \ z = z \land \big((x + x) = z\big) \to \big((2 + 2) = z \lor y > z\big)$

1.	$\neg\big(y > (x+x) \vee x = 2\big) \rightarrow \neg(y = y)$	premise
2.	$y = y$	$=$i
3.	$\neg\neg(y = y)$	$\neg\neg$i 2
4.	$\neg\neg\big(y > (x+x) \vee x = 2\big)$	MT 1,3
5.	$y > (x+x) \vee x = 2$	$\neg\neg$e 4
6.	$z = z \wedge ((x+x) = z)$	assumption
7.	$(x+x) = z$	$\wedge e_2$ 6
8.	$y > (x+x)$	assumption
9.	$y > z$	$=$e 7,8
10.	$(2+2) = z \vee y > z$	$\vee i_2$ 9
11.	$x = 2$	assumption
12.	$(2+2) = z$	$=$e 7,11
13.	$(2+2) = z \vee y > z$	$\vee i_1$ 12
14.	$(2+2) = z \vee y > z$	$\vee$e 5,8-10,11-13
15.	$z = z \wedge \big((x+x) = z\big) \rightarrow \big((2+2) = z \vee y > z\big)$	$\rightarrow$i 6-14