

Lecture Notes for

# Logic and Computability

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# Table of Contents

<b>2</b>	<b>Natural Deduction for Propositional Logic</b>	<b>1</b>
2.1	Rules for natural deduction . . . . .	2
2.2	Soundness and Completeness of ND . . . . .	14
2.2.1	Invalid Sequents . . . . .	14
2.3	Exercises . . . . .	16
2.3.1	Examples . . . . .	16
2.3.2	Solutions . . . . .	17
2.3.3	Self-Evaluation . . . . .	20



# 2

## Natural Deduction for Propositional Logic

We started the first chapter by using our common sense to conclude new knowledge from given knowledge (we concluded that “there were taxis at the airport” and that “John has his sun creme with him”). Our goal is to perform this reasoning formally and automatically. Natural deduction is a calculus for reasoning about propositions so that we can establish the validity of arguments. Therefore, natural deduction defines a *set of rules* each of which allows us to draw a conclusion given a certain arrangement of premises. By successively applying these rules, we are able to *infer* a conclusion from a set of premises.

### Sequents

Our goal is to apply proof rules to a set of given formulas –*the premises*– to eventually obtain a new formula –*the conclusion*. Formally, we write:

$$\phi_1, \phi_2, \dots, \phi_n \vdash \psi$$

We call this expression a *sequent*. We say that the *premises* (formula on the left side) *entail* the *conclusion* (formula on the right). *A sequent is valid if a proof for it can be found.*

**Example.** The sequent for the illustration example in Chapter 1 is:

$$p \wedge \neg q \rightarrow r, \neg r, p \vdash q$$

## 2.1 Rules for natural deduction

For each of the connectives, there is one or more rules to introduce it and one or more rules to eliminate it.

### The ‘AND-Introduction’ Rule

First, we consider the rule for introducing a conjunction, called the *and-introduction-rule*. Given the two premises  $\varphi$  and  $\psi$ , the rule allows us to conclude  $\varphi \wedge \psi$ . We write:

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \wedge i$$

Above the line we write the two premises  $\varphi$  and  $\psi$  of the rule. Below the line we write the conclusion  $\varphi \wedge \psi$ . To the right of the line, we state the name of the rule; ‘and-introduction’ is abbreviated by  $\wedge i$ .

The intuition of the rule is quite easy to understand. If we have two formulas that are known to be true separately (the premises), then we can conclude that the conjunction of the two premises must also be true. Simple illustration:

- Butterflies can fly (Premise)
- Bunnies can hop (Premise)
- Therefore: Butterflies can fly *and* bunnies can hop. ( $\wedge i$  of line 1 and 2)
- Therefore: Bunnies can hop *and* butterflies can fly. ( $\wedge i$  of line 2 and 1)

### Construction of a Natural Deduction Proof

Next, we discuss how to construct a proof using the natural deduction rules to show that a given sequent is valid.

**Example.** Give the proof for the sequents  $p, q \vdash p \wedge q$  and  $p, q \vdash q \wedge p$ .

$$p, q \vdash p \wedge q$$

1.  $p$       prem.
2.  $q$       prem.
3.  $p \wedge q$     $\wedge i$  1,2

$$p, q \vdash q \wedge p$$

1.  $p$       prem.
2.  $q$       prem.
3.  $q \wedge p$     $\wedge i$  2,1

Each line of the proof consists of the *line number*, a *formula*, and *the reason for having the formula*. We start the proof by writing down the premises, leaving a gap, and writing the conclusion in the end. The task is to apply the rules such that we fill the gap. In this case, we only need to write down, that we applied the  $\wedge i$  rule, once combining line 1 and line 2, and once in the reverse order, to justify the conclusion.

### The ‘AND-Elimination’ Rule

Given the premise  $\varphi \wedge \psi$ , the elimination rules allows us to conclude  $\varphi$  as well as  $\psi$ . We write:

$$\frac{\varphi \wedge \psi}{\varphi} \wedge e1 \qquad \frac{\varphi \wedge \psi}{\psi} \wedge e2$$

The rule  $\wedge e1$  is used to derive the first subformula, the rule  $\wedge e2$  is used to derive the second sub-formula. Intuitively, if a conjunction is known to be true, each sub-formula must also be true. Intuitive illustration:

- The earth is a planet and the sun is a star. (Premise)
- Therefore: The earth is a planet. ( $\wedge e1$  line 1)
- Therefore: The sun is a star. ( $\wedge e2$  line 1)

**Example.** Give the proof for the sequents  $p \wedge q \vdash p$  and  $p \wedge q \vdash q$ .

$p \wedge q \vdash p$ <ol style="list-style-type: none"> <li>1. <math>p \wedge q</math>    prem.</li> <li>2. <math>p</math>        <math>\wedge e1</math> 1</li> </ol>	$p \wedge q \vdash q$ <ol style="list-style-type: none"> <li>1. <math>p \wedge q</math>    prem.</li> <li>2. <math>q</math>        <math>\wedge e2</math> 1</li> </ol>
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**Example.** Give the proof for the sequent  $p \wedge q, r \vdash q \wedge r$ .

1.  $p \wedge q$     prem.
2.  $r$         prem.
3.  $q$          $\wedge e2$  1
4.  $q \wedge r$     $\wedge i$  3,2

**Example.** Give the proof for the sequent  $(p \wedge q) \wedge r, s \wedge t \vdash q \wedge s$ .

1.  $(p \wedge q) \wedge r$     prem.
2.  $s \wedge t$         prem.
3.  $p \wedge q$          $\wedge e1$  1
4.  $q$              $\wedge e2$  3
5.  $s$              $\wedge e1$  2
6.  $q \wedge s$          $\wedge i$  4,5

In order to form the conclusion, the propositions  $q$  and  $s$  are needed.  $q$  can be extracted from the first premise. Note: a natural deduction rule can only be applied on the *top-level connective* of a formula. Hence, we need to apply the  $\wedge e$  rule once to get  $p \wedge q$ , and then a second time to get  $q$ . Furthermore, we need the propositional atom  $s$  to form the conclusion. We get  $s$  from the second premise by eliminating the  $t$ . Finally,  $q$  and  $s$  can be connected using the  $\wedge i$  rule to form the conclusion.

### The ‘Double-Negation-Introduction’ Rule

If a formula  $\varphi$  holds, also  $\neg\neg\varphi$  must be true, since they are equivalent. The rule looks as following.

$$\frac{\varphi}{\neg\neg\varphi} \neg\neg\text{i}$$

Intuitively, the sentence “The ocean is salty” is the same as saying “It is not true that the ocean is not salty.”

**Example.** Give the proof for the sequent  $p \wedge q \vdash \neg\neg p$ .

1.  $p \wedge q$     prem.
2.  $p$          $\wedge\text{e1 1}$
3.  $\neg\neg p$      $\neg\neg\text{i 2}$

### The ‘Double-Negation-Elimination’ Rule

The rule is written as follows.

$$\frac{\neg\neg\varphi}{\varphi} \neg\neg\text{e}$$

Same argument as before, the two formulas are equivalent. If it is true that “Great Britain is not not a monarchy”, then we can follow that “Great Britain is a monarchy”.

**Example.** Give the proof for the sequent  $\neg\neg p \wedge \neg\neg q \vdash p \wedge q$ .

1.  $\neg\neg p \wedge \neg\neg q$     prem.
2.  $\neg\neg p$              $\wedge\text{e1 1}$
3.  $\neg\neg q$              $\wedge\text{e2 1}$
4.  $p$                  $\neg\neg\text{e 2}$
5.  $q$                  $\neg\neg\text{e 3}$
6.  $p \wedge q$          $\wedge\text{i 4,5}$

**Example.** Give the proof for the sequent  $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$ .

1.  $p$                 prem.
2.  $\neg\neg(q \wedge r)$     prem.
3.  $\neg\neg p$              $\neg\neg\text{i 1}$
4.  $q \wedge r$          $\neg\neg\text{e 2}$
5.  $r$                  $\wedge\text{e2 4}$
6.  $\neg\neg p \wedge r$      $\wedge\text{i 3,5}$



### The ‘Implication-Elimination’ Rule

The implication-elimination rule states that, if we know that  $\varphi$  holds and we know that  $\varphi \rightarrow \psi$ , we can conclude that  $\psi$  holds.

$$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \rightarrow e$$

Intuitively, if we know that it is true that “It is snowing”, and “If it is snowing then it is cold”, then we can conclude that “It is cold”.

**Example.** Give the proof for the sequent  $\neg\neg p, p \rightarrow q \vdash \neg\neg q$ .

1.  $\neg\neg p$       prem.
2.  $p \rightarrow q$     prem.
3.  $p$              $\neg\neg e$  1
4.  $q$              $\rightarrow e$  1,2
5.  $\neg\neg q$         $\neg\neg i$  4

**Example.** Give the proof for the sequent  $p \wedge \neg a, p \wedge \neg a \rightarrow q \vee b \vdash q \vee b$ .

1.  $p \wedge \neg a$             prem.
2.  $p \wedge \neg a \rightarrow q \vee b$     prem.
3.  $q \vee b$                  $\rightarrow e$  1,2

**Example.** Give the proof for the sequent  $p, p \rightarrow q, p \rightarrow (q \rightarrow r) \vdash r$ .

1.  $p$                     prem.
2.  $p \rightarrow q$         prem.
3.  $p \rightarrow (q \rightarrow r)$     prem.
4.  $q \rightarrow r$          $\rightarrow e$  1,3
5.  $q$                      $\rightarrow e$  1,2
6.  $r$                      $\rightarrow e$  4,5

### The ‘Modus-Tollens’ Rule (MT)

Before discussing the implication-introduction rule, let us consider a derived rule from the implication-elimination rule called modus tollens. If it holds that  $\varphi \rightarrow \psi$  and  $\neg\psi$  are true, then we can conclude  $\neg\varphi$ .

$$\frac{\varphi \rightarrow \psi \quad \neg\psi}{\neg\varphi} \text{MT}$$

Intuitive argumentation. The following is true: “If the sun is shining it is daytime” and “It is not daytime”. Therefore, we can conclude using modus tollens that “The sun is not shining”.

**Example.** Give the proof for the sequent  $\neg p \rightarrow q, \neg q \vdash p$ .

1.  $\neg p \rightarrow q$     prem.
2.  $\neg q$             prem.
3.  $\neg\neg p$         MT 1,2
4.  $p$               $\neg\neg$ e 3

**Example.** Give the proof for the sequent  $p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q$ .

1.  $\neg p \rightarrow (q \rightarrow r)$     prem.
2.  $\neg p$                     prem.
3.  $\neg r$                     prem.
4.  $q \rightarrow r$              $\rightarrow$ e 1,2
5.  $\neg q$                     MT 4,3

### The ‘Implication-Introduction’ Rule

The  $\rightarrow$ i rule says that in order to prove  $\varphi \rightarrow \psi$ , we make a temporary assumption  $\varphi$  and then prove  $\psi$ . The rule is formally written as:

$$\frac{\begin{array}{|l} \varphi \text{ ass.} \\ \vdots \\ \psi \end{array}}{\varphi \rightarrow \psi} \rightarrow\text{i}$$

Lets assume that we want to prove the sequent  $p \rightarrow q, q \rightarrow r \vdash p \rightarrow r$ . To prove this sequent, we *temporarily assume* that  $p$  holds. *Under the assumption* that  $p$  holds, we can derive from the first premise that  $q$  holds, and using  $q$  we can derive that  $r$  holds from the second premise. Thus, by *assuming* that  $p$  holds, we can *imply* that  $r$  holds, which we express symbolically by  $p \rightarrow r$ . The prove is given below.

1.  $p \rightarrow q$     prem.
2.  $q \rightarrow r$     prem.
3. 

$p$	ass.
$q$	$\rightarrow$ e 3,1
$r$	$\rightarrow$ e 3,2
4.  $q$              $\rightarrow$ e 3,1
5.  $r$              $\rightarrow$ e 3,2
6.  $p \rightarrow r$      $\rightarrow$ i 2-3

The *assumption box* in the proof defines the scope of the temporary assumption  $p$ . By applying other rules, we can derive new formulas within the box. But everything that we derive inside of the box still depends on the assumption of  $p$ . Only by applying the  $\rightarrow i$  rule, we are allowed to conclude  $p \rightarrow r$ . We will introduce additional rules that uses boxes. *It is important that the line immediately following a closed box has to match the pattern of the conclusion of the rule that uses the box.* For the  $\rightarrow i$  rule this means that we have to continue after the box with  $\varphi \rightarrow \psi$ . Within the box,  $\varphi$  is the formula in the first line and  $\psi$  the formula of the last line.

**Example.** Give the proof for the sequent  $p \rightarrow q \vdash \neg q \rightarrow \neg p$ .

1.	$p \rightarrow q$	prem.
2.	$\neg q$	ass.
3.	$\neg p$	MT 1,2
4.	$\neg q \rightarrow \neg p$	$\rightarrow i$ 2-3

**Example.** Give the proof for the sequent  $\neg q \rightarrow \neg p \vdash p \rightarrow \neg \neg q$ .

1.	$\neg q \rightarrow \neg p$	prem.
2.	$p$	ass.
3.	$\neg \neg p$	$\neg \neg i$ 2
4.	$\neg \neg q$	MT 1,3
5.	$p \rightarrow \neg \neg q$	$\rightarrow i$ 2-4

**Example.** Give the proof for the sequent  $p \wedge q \rightarrow r \vdash p \rightarrow (q \rightarrow r)$ .

1.	$p \wedge q \rightarrow r$	prem.
2.	$p$	ass.
3.	$q$	ass.
4.	$p \wedge q$	$\wedge i$ 2,3
5.	$r$	$\rightarrow e$ 4,1
6.	$q \rightarrow r$	$\rightarrow i$ 3-5
7.	$p \rightarrow (q \rightarrow r)$	$\rightarrow i$ 2-6

**Example.** Give the proof for the sequent  $p \rightarrow (q \rightarrow r) \vdash p \wedge q \rightarrow r$ .

1.	$p \rightarrow (q \rightarrow r)$	prem.
2.	$p \wedge q$	ass.
3.	$p$	$\wedge e1$ 2
4.	$q$	$\wedge e2$ 2
5.	$q \rightarrow r$	$\rightarrow e$ 3,1
6.	$r$	$\rightarrow e$ 4,5
7.	$p \wedge q \rightarrow r$	$\rightarrow i$ 2-6

### The ‘Disjunction-Introduction’ Rule

If we know that  $\varphi$  holds, we can derive that  $\varphi \vee \psi$  holds and that  $\psi \vee \varphi$  holds. This is true for any  $\psi$ . The rule is formulated as follows:

Formally the rules are written as:

$$\frac{\varphi}{\varphi \vee \psi} \vee i1 \qquad \frac{\varphi}{\psi \vee \varphi} \vee i2$$

**Example.** Give the proofs for the sequent  $p \vdash (q \rightarrow r \wedge s) \vee p$ .

1.	$p$	prem.
2.	$(q \rightarrow r \wedge s) \vee p$	$\vee i2$ 1

### The ‘Disjunction-Elimination’ Rule

From a given formula  $\varphi \vee \psi$ , we want to proof some other formula  $\chi$ . We only know that  $\varphi$  *or*  $\psi$  holds. It could be that both of them are true, but it could also be that only  $\psi$  is true, or only  $\varphi$  is true. Since we don’t know which sub-formula is true, we have to give two separate proofs:

- First box: We assume  $\varphi$  is true and need to find a proof for  $\chi$ .
- Second box: We assume  $\psi$  is true and need to find a proof for  $\chi$ .

Only if we can prove  $\chi$  in the first and in the second box, then we can conclude that  $\chi$  holds also outside of the box.

The  $\vee e$  rules says that we can only derive  $\chi$  from  $\varphi \vee \psi$  if we can derive  $\chi$  from the assumption  $\varphi$  as well as from the assumption  $\psi$ . Formally the rule is written as:

$$\frac{\varphi \vee \psi \quad \begin{array}{|c|} \hline \varphi \text{ ass.} \\ \vdots \\ \chi \\ \hline \end{array} \quad \begin{array}{|c|} \hline \psi \text{ ass.} \\ \vdots \\ \chi \\ \hline \end{array}}{\chi} \vee e$$

**Example.** Give the proof for the sequent  $p \vee q \vdash q \vee p$ .

$$p \vee q \vdash q \vee p$$

1.	$p \vee q$	prem.
2.	$p$	ass.
3.	$q \vee p$	$\vee i 2$
4.	$q$	ass.
5.	$q \vee p$	$\vee i 4$
6.	$q \vee p$	$\vee e 1,2-3,4-5$

**Example.** Give the proof for the sequent  $q \rightarrow r \vdash (p \vee q) \rightarrow (p \vee r)$ .

1.	$q \rightarrow r$	prem.
2.	$p \vee q$	ass.
3.	$p$	ass.
4.	$p \vee r$	$\vee i 1 2$
5.	$q$	ass.
6.	$r$	$\rightarrow e 5,1$
7.	$p \vee r$	$\vee i 6$
8.	$p \vee r$	$\vee e 2,3-4,5-7$
9.	$p \vee q \rightarrow (p \vee r)$	$\rightarrow i 2-8$

**Example.** Give the proof for the sequent:  
 $p \wedge (q \vee r), p, (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$ .

1.	$p \wedge (q \vee r)$	prem.
2.	$p$	prem.
3.	$(q \vee r)$	prem.
4.	$q$	ass.
5.	$p \wedge q$	$\wedge i 2,4$
6.	$(p \wedge q) \vee (p \wedge r)$	$\vee i 1 5$
7.	$r$	ass.
8.	$p \wedge r$	$\wedge i 2,7$
9.	$(p \wedge q) \vee (p \wedge r)$	$\vee i 2 8$
10.	$(p \wedge q) \vee (p \wedge r)$	$\vee e 3, 4-6, 7-9$

### The 'Copy'-Rule

The copy rules allows us to repeat any formula that we have already proven. This is helpful when we need to conclude a box with a formula that we have already proven outside of the box. In this case, the formula can simply be copied into the box which can then be closed.

**Example.** Give the proof for the sequent  $p \vdash q \rightarrow (p \vee t)$ .

1.	$p$	prem.
2.	$q$	ass.
3.	$p$	copy 1
4.	$p \vee t$	$\vee$ i1 3
5.	$q \rightarrow (p \vee t)$	$\rightarrow$ i 2-4

**Definition.** Formulas in propositional logic  $\varphi$  with valid sequent  $\vdash \varphi$  are called *theorems*.

**Example.** Give the proof for the sequent  $\vdash p \rightarrow q \rightarrow p$ .

1.	$p$	ass.
2.	$q$	ass.
3.	$p$	copy 1
4.	$q \rightarrow p$	$\rightarrow$ i 2-3
5.	$p \rightarrow q \rightarrow p$	$\rightarrow$ i 1-4

### The 'Contradiction-Elimination' Rule

**Definition.** A *contradiction* is an expressions of the form  $\varphi \wedge \neg\varphi$  or  $\neg\varphi \wedge \varphi$ , where  $\varphi$  is any formula.

Examples for contradictions are:  $r \wedge \neg r$  and  $(p \rightarrow q) \wedge \neg(p \rightarrow q)$ .

Any formula can be derived from a contradiction. Therefore, the proof rule for contradiction elimination looks as follows.

$$\frac{\perp}{\varphi} \perp e$$

The rule expresses that we can derive anything from a contradiction. Lets say, that our two premises say "Sunflowers are plants" and "Sunflowers are not plants". These two premises cannot be true at the same time, and we can infer a contradiction. From the contradiction we can infer anything, like e.g., *Therefore*, "drinking energy drinks helps you sleep better." If a formula on the left hand side of an entailment relation is false, the entire sequent is trivially true.

### The 'Negation-Elimination' Rule

We use the negation-elimination rule to derive a contradiction from the given formulas  $\varphi$  and  $\neg\varphi$ . Formally the rule is written as:

$$\frac{\varphi \quad \neg\varphi}{\perp} \neg e$$

**Example.** Give the proof for the sequent  $\neg p, p \vdash q$ .

1.  $\neg p$  prem.
2.  $p$  prem.
3.  $\perp$   $\neg e$  2,1
4.  $q$   $\perp e$  3

**Example.** Give the proof for the sequent  $p \vee \neg q \vdash q \rightarrow (p \vee r)$ .

1.  $p \vee \neg q$  prem.
2.  $q$  ass.
3.  $p$  ass.
4.  $p \vee r$   $\vee i$  3
5.  $\neg q$  ass.
6.  $\perp$   $\neg e$  2,5
7.  $p \vee r$   $\perp e$  6
8.  $p \vee r$   $\vee e$  1,3-4,5-7
9.  $q \rightarrow (p \vee r)$   $\rightarrow i$  2-8

### The 'Negation-Introduction' Rule

Lets assume that we make an assumption which gets us a contradiction. If this is the case, our assumption must be false. The  $\neg i$  rule captures this intuition:

$$\frac{\begin{array}{|l} \varphi \text{ ass.} \\ \vdots \\ \perp \end{array}}{\neg \varphi} \neg i$$

**Example.** Give the proof for the sequent  $p \rightarrow \neg q, q \vdash \neg p$ .

1.  $p \rightarrow \neg q$  prem.
2.  $q$  prem.
3.  $p$  ass.
4.  $\neg q$   $\rightarrow e$  3,1
5.  $\perp$   $\neg e$  2,4
6.  $\neg p$   $\neg i$  3-5

**Example.** Give the proof for the sequent  $p \rightarrow \neg p \vdash \neg p$ .

1.	$p \rightarrow \neg p$	prem.
2.	$p$	ass.
3.	$\neg p$	$\rightarrow e$ 1,2
4.	$\perp$	$\neg e$ 2,3
5.	$\neg p$	$\neg i$ 2-4

**Example.** Give the proof for the sequent  $p \wedge \neg q \rightarrow r, \neg r, p \vdash q$ .

1.	$p \wedge \neg q \rightarrow r$	prem.
2.	$\neg r$	prem.
3.	$p$	prem.
4.	$\neg q$	ass.
5.	$p \wedge \neg q$	$\wedge i$ 3,4
6.	$r$	$\rightarrow e$ 1,5
7.	$\perp$	$\neg e$ 6,2
8.	$\neg\neg q$	$\neg i$ 2-4
9.	$q$	$\neg\neg e$ 8

### The 'Proof-by-Contradiction' Rule (PCB)

Another handy derived-rule is called the proof-by-contradiction rule (PBC). It is very similar to the  $\neg i$  rule. The rule states that if from  $\neg\varphi$  we obtain a contradiction, then we are allowed to conclude  $\varphi$ :

$\neg\varphi$ ass. $\vdots$ $\perp$	$\frac{\quad}{\varphi}$	PBC
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**Example.** Give the proof for the sequent  $\neg p \rightarrow \neg q, q \vdash p$ .

1.	$\neg p \rightarrow \neg q$	prem.
2.	$q$	prem.
3.	$\neg p$	ass.
4.	$\neg q$	$\rightarrow e$ 3,1
5.	$\perp$	$\neg e$ 2,4
6.	$p$	PBC 3-5

### The 'Law-of-the-Excluded-Middle' Rule (LEM)

The LEM simply says that  $\varphi \vee \neg\varphi$  is true. For every formula  $\varphi$  it holds that it is either true or false, therefore the sequent  $\vdash \varphi \vee \neg\varphi$  is valid.



$$\frac{}{\varphi \vee \neg\varphi} \text{LEM}$$

**Example.** Give the proof for the sequent  $p \vdash q \wedge r \vee \neg(q \wedge r)$ .

1.  $p$                       prem.
2.  $q \wedge r \vee \neg(q \wedge r)$     LEM

**Example.** Give the proof for the sequent  $p \rightarrow q \vdash \neg p \vee q$  using LEM.

1.  $p \rightarrow q$     prem.
2.  $\neg p \vee p$     LEM
3. 

$\neg p$	ass.
----------	------
4. 

$\neg p \vee q$	$\vee i1$ 3
-----------------	-------------
5. 

$p$	ass.
-----	------
6. 

$q$	$\rightarrow e$ 1,5
-----	---------------------
7. 

$\neg p \vee q$	$\vee i2$ 6
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8.  $\neg p \vee q$      $\vee e$  2,3-5,5-7

### Tips for making your own Natural Deduction Proof

- **Start a proof.** At the top of your page write the premises, at the bottom write the conclusion.
- **Work in both directions to fill the gap.** Work from the top to the bottom by working with the premises, and simultaneously work upwards by using the conclusion.
- **Look first at the conclusion.** If the conclusion is of the form  $\varphi \rightarrow \psi$ , then immediately apply  $\rightarrow i$ . You still have to fill the gap in the box, but you have an extra assumption to work with and a simpler conclusion you try to reach. Similar, if your conclusion is of the form  $\neg\varphi$ , apply  $\neg i$  to make your life easier.
- **Assumption boxes.** At any time you can introduce a formula as assumption, by choosing a proof rule that opens the box. The box defines the scope of the assumption. By opening a box you introduce an assumption. But don't forget, you have to close the box precisely as defined by the applied proof rule.
- **What rule should you apply?** The rules  $\rightarrow i$  and  $\neg i$  make your life easier, apply them whenever you can. There is no easy recipe for when to use the other rules. The best way to get the hang of it is doing many proofs by yourself.

## 2.2 Soundness and Completeness of ND

### Soundness

*Natural deduction for propositional logic is sound. Therefore, any sequent that can be proven is a correct semantic entailment.*

$$\phi_1, \phi_2, \dots, \phi_n \vdash \psi \quad \Rightarrow \quad \phi_1, \phi_2, \dots, \phi_n \models \psi$$

So, if we have proven with natural deduction that a sequent  $\phi_1, \phi_2, \dots, \phi_n$  is valid, then for all valuations in which all premises  $\phi_1, \phi_2, \dots, \phi_n$  evaluate to *true*,  $\psi$  evaluates to *true* as well.

From soundness also follows that *if the semantic entailment relation does not hold, the sequent cannot be proven using natural deduction.*

$$\phi_1, \phi_2, \dots, \phi_n \not\models \psi \quad \Rightarrow \quad \phi_1, \phi_2, \dots, \phi_n \not\vdash \psi$$

### Completeness

*Natural deduction for propositional logic is sound. Therefore, any sequent that is a correct semantic entailment can be proven.*

$$\phi_1, \phi_2, \dots, \phi_n \models \psi \quad \Rightarrow \quad \phi_1, \phi_2, \dots, \phi_n \vdash \psi$$

From completeness also follows that *if a sequent is not provable that means it is no correct semantic entailment.*

$$\phi_1, \phi_2, \dots, \phi_n \not\vdash \psi \quad \Rightarrow \quad \phi_1, \phi_2, \dots, \phi_n \not\models \psi$$

### Corollary: Soundness and Completeness

*Natural deduction for propositional logic is sound and complete.*

Let  $\phi_1, \phi_2, \dots, \phi_n, \psi$  be formulas of propositional logic. Then  $\phi_1, \phi_2, \dots, \phi_n \models \psi$  is holds if and only if the sequent  $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$  is valid.

#### 2.2.1 Invalid Sequents

To show that a sequent is invalid, we need to find a *counter example*. A counter example is a model, that *satisfies all premises but falsifies the conclusion*.

**Example.** Show that the sequent  $p \wedge q \vdash \neg p$  is *not* valid by finding a counter-example.

The model

$$\mathcal{M} : p = T, q = T$$

is a counter example, since it satisfies the premise, i.e.,  $\mathcal{M} \models p \wedge q$ , and it does not satisfy the conclusion, i.e.,  $\mathcal{M} \not\models \neg p$ .

**Example.** Find two counter-examples for the sequent  $p \vee q \vdash p \wedge q$ .

$$\mathcal{M} : p = T, q = F$$

$$\mathcal{M} \models p \vee q, \mathcal{M} \not\models p \wedge q$$

$$\text{Therefore, } p \vee q \not\models p \wedge q$$

$$\mathcal{M} : p = F, q = T$$

$$\mathcal{M} \models p \vee q, \mathcal{M} \not\models p \wedge q$$

$$\text{Therefore, } p \vee q \not\models p \wedge q$$

## 2.3 Exercises

### 2.3.1 Examples

Prove the following examples using natural deduction or disprove them by finding a counterexample. Pay attention to correct operator precedence.

**Example 1.**  $p \wedge q \rightarrow r, p, q \vdash r$

**Example 2.**  $(\neg q \wedge p) \wedge r \vdash p \vee q$

**Example 3.**  $\neg\neg r \rightarrow \neg\neg\neg p, \neg\neg(p \wedge r) \vdash s$

**Example 4.**  $r \rightarrow s \rightarrow t \vdash (r \rightarrow t) \rightarrow s$

**Example 5.**  $\vdash r \rightarrow (r \rightarrow s) \rightarrow r$

**Example 6.**  $(a \vee \neg b) \wedge (\neg a \vee b) \vdash \neg(a \wedge b)$

**Example 7.**  $\neg(a \vee b) \rightarrow c, \neg c \vdash a \vee b$

(a) Solve with MT.

(b) Solve without MT.

**Example 8.**  $(p \vee q) \wedge r \vdash p \vee (q \vee s)$

**Example 9.**  $a \rightarrow b \vdash (\neg a \rightarrow c) \rightarrow \neg c \rightarrow b$

**Example 10.**  $a \vee \neg b, b \vee \neg c \vdash a \vee c \rightarrow b$

**Example 11.**  $a \vee \neg b, b \vee \neg c \vdash b \rightarrow a \vee c$

**Example 12.**  $p \rightarrow q \vdash \neg p \vee q$

## 2.3.2 Solutions

**Solution 1.** This sequent is provable.

1.  $(p \wedge q) \rightarrow r$  prem.
2.  $p$  prem.
3.  $q$  prem.
4.  $p \wedge q$   $\wedge$ i 2,3
5.  $r$   $\rightarrow$ e 4,1

**Solution 2.** This sequent is provable.

1.  $(\neg q \wedge p) \wedge r$  prem.
2.  $\neg q \wedge p$   $\wedge$ e1 1
3.  $p$   $\wedge$ e2 2
4.  $p \vee q$   $\vee$ i1 3

**Solution 3.** This sequent is provable.

1.  $\neg\neg r \rightarrow \neg\neg\neg p$  prem.
2.  $\neg\neg(p \wedge r)$  prem.
3.  $p \wedge r$   $\neg\neg$ e 2
4.  $p$   $\wedge$ e1 3
5.  $r$   $\wedge$ e2 3
6.  $\neg\neg r$   $\neg\neg$ i 5
7.  $\neg\neg\neg p$   $\rightarrow$ e 6,1
8.  $\neg p$   $\neg\neg$ e 7
9.  $\vdash$   $\neg$ e 4,8
10.  $s$   $\perp$ e 9

**Solution 4.** This sequent is not provable.

$\mathcal{M} : r = F, s = F, t = T$

$\mathcal{M} \models r \rightarrow s \rightarrow t \vdash (r \rightarrow t) \rightarrow s, \mathcal{M} \not\models s$

$r \rightarrow s \rightarrow t \vdash (r \rightarrow t) \rightarrow s \not\models s$

**Solution 5.** This sequent is provable.

1. 

$r$	ass.
-----	------
2. 

$r \rightarrow s$	ass.
-------------------	------
3. 

$r$	copy 1
-----	--------
4.  $(r \rightarrow s) \rightarrow r$   $\rightarrow$ i 2-3
5.  $\vdash r \rightarrow ((r \rightarrow s) \rightarrow r)$   $\rightarrow$ i 1-4
- 6.

**Solution 6.** This sequent is not provable.

$\mathcal{M} : a = T, b = T$

$\mathcal{M} \models (a \vee \neg b \wedge (\neg a \vee b)), \mathcal{M} \not\models \neg(a \wedge b)$

$(a \vee \neg b \wedge (\neg a \vee b)) \not\models \neg(a \wedge b)$

**Solution 7.** This sequent is provable.

(a) Without MT.

1.	$\neg(a \vee b) \rightarrow c$	prem.
2.	$\neg c$	prem.
3.	$\neg(a \vee b)$	ass.
4.	$p$	$\rightarrow e$ 3,1
5.	$\perp$	$\neg e$ 4,2
6.	$a \vee b$	PBC 3-5

(b) With MT.

1.	$\neg(a \vee b) \rightarrow c$	prem.
2.	$\neg c$	prem.
3.	$\neg\neg(a \vee b)$	MT 1,2
4.	$a \vee b$	$\neg\neg e$ 3

**Solution 8.** This sequent is provable.

1.	$(p \vee q) \wedge r$	prem.
2.	$p \vee q$	$\wedge e1$ 1
3.	$p$	ass.
4.	$p \vee (q \vee s)$	$\vee i1$ 3
5.	$q$	ass.
6.	$q \vee s$	$\vee i1$ 5
7.	$p \vee (q \vee s)$	$\vee i2$ 6
8.	$p \vee (q \vee s)$	$\vee e$ 2,3-4,5-6

**Solution 9.** This sequent is provable.

1.	$a \rightarrow b$	prem.
2.	$\neg a \rightarrow c$	ass.
3.	$\neg c$	ass.
4.	$\neg\neg a$	MT 2,3
5.	$a$	$\neg\neg e$ 4
6.	$b$	$\rightarrow e$ 5,1
7.	$\neg c \rightarrow b$	$\rightarrow i$ 3-6
8.	$(\neg a \rightarrow c) \rightarrow (\neg c \rightarrow b)$	$\rightarrow i$ 2-7

**Solution 10.** This sequent is not provable.

$$\mathcal{M} : a = T, b = F, c = F$$

$$\mathcal{M} \models (a \vee \neg b), \mathcal{M} \models (b \vee \neg c), \mathcal{M} \not\models a \vee c \rightarrow b$$

$$(a \vee \neg b), (b \vee \neg c), a \vee c \rightarrow b$$

**Solution 11.** This sequent is provable.

1.	$a \vee \neg b$	prem.
2.	$b \vee \neg c$	prem.
3.	$b$ ass.	
4.	$a$ ass.	
5.	$a \vee c$ $\vee$ i1 4	
6.	$\neg b$ ass.	
7.	$\perp$ $\neg$ e 3,6	
8.	$a \vee c$ $\perp$ e 7	
9.	$a \vee c$ $\vee$ e 1,4-5,6-8	
10.	$b \rightarrow (a \vee c)$	$\rightarrow$ i 3-9

**Solution 12.** This sequent is provable.

1.	$(p \rightarrow q)$	prem.
2.	$p \vee \neg p$	LEM
3.	$p$ ass.	
4.	$q$ $\rightarrow$ e 2,1	
5.	$\neg p \vee q$ $\vee$ i2 3	
6.	$\neg p$ ass.	
7.	$\neg p \vee q$ $\vee$ i2 6	
8.	$\neg p \vee q$	$\vee$ e 2,3-4,5-7

### 2.3.3 Self-Evaluation

*These examples will be solved online during class.*

For each of the following sequents, either provide a natural deduction proof, or a counter-example that proves the sequent invalid.

For proofs, clearly indicate which rule, and what assumptions/premises/intermediate results you are using in each step. Also clearly indicate the scope of any boxes you use.

For counterexamples, give a complete model. Show that the model satisfies the premise(s) of the sequent in question, but does not satisfy the respective conclusion.

- Example 1.**
- (a) If I press the button, the window opens.  
I pressed the button.  
Therefore, the window is open.
  - (b) If I press the button, the window opens.  
The window is open.  
Therefore, I pressed the button.
  - (c) (Solve without using the Modus Tollens)  
If I press the button, the window opens.  
The window is not open.  
Therefore, I didn't press the button.

**Example 2.**  $(p \rightarrow r) \wedge (q \rightarrow s) \vdash (p \rightarrow q) \rightarrow (r \wedge s)$

**Example 3.**  $p \rightarrow (q \vee r), (q \rightarrow s), (r \rightarrow s) \vdash (p \rightarrow s)$

**Example 4.**  $\vdash (p \rightarrow q) \vee (q \rightarrow r)$

**Example 5.**  $\neg q \vee \neg p \vdash \neg(q \wedge p)$

**Example 6.**  $p \vee q, \neg p \vee r \vdash q \vee r$



- Example 7.** Explain the  $\vee$ -elimination rule of the natural deduction calculus. Why does this rule require two boxes?
- Example 8.** “Natural deduction for propositional logic is *sound* and *complete*.” Explain in your own words what this means.
- Example 9.** Natural deduction for propositional logic is sound and complete. In the following list, mark each statement with either **S**, **C**, **B**, or **N**, depending on whether the corresponding statement follows from **S**oundness, **C**ompleteness, **B**oth, or **N**either. (Note: If a statement is in itself factually wrong, or has nothing to do with soundness and completeness, mark it **N**, since it follows from neither soundness nor completeness.)
- A sequent has a proof if and only if it is semantically correct.
  - If a model satisfies the premise(s) of a given sequent, but does not satisfy the conclusion of the sequent, it is not possible to construct a proof for the sequent.
  - There is no correct sequent for which there is no proof.
  - Every sequent has a proof.
  - An incorrect sequent does not have a proof.
  - Every propositional formula is either valid or not valid.
  - If all models that satisfy the premise(s) of a given sequent also satisfy the conclusion of the sequent, there exists a proof for the sequent.

### Declaration of Sources

Chapter 2 was based on the following book.

Michael Huth, Mark Dermot Ryan: *Logic in Computer Science: Modelling and Reasoning about Systems*. June 2004. Cambridge University Press. ISBN:978-0-521-54310-1