Propositional Logic

Syntax, Semantics, Models, Applications

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Motivation

- Basis for further topics
- Syntax
  - Construct formulas
- Semantics
  - Understand formulas
- See usage examples
Outline

- Syntax
  - Symbols
  - Grammar

- Semantics
  - Meaning
  - Models
  - Truth Tables
  - Validity, Satisfiability, Semantic Entailment & Equivalence

- Applications
Learning Targets
Syntax

- Explain syntax of propositional formulas
  - Based on examples

- Name elements/symbols/connectives of propositional formulas

- Draw parse tree of propositional formulas
Learning Targets
Semantics

- Explain semantics of prop. formula
  - Based on a model
- Construct and explain truth table of prop. formula
- Decide validity, satisfiability, semantic entailment/equivalence of prop. formula(s)
  - Using truth tables
- Explain validity, satisfiability, semantic entailment/equivalence
  - Using examples
- Model declarative sentences as prop. formula
  - As detailed as possible
Learning Targets

Applications

- Name and explain examples for usage of prop. logic to solve problems

- Solve suitable problems by using prop. logic
  - Reduction to “Classical questions”
Operator Precedence

- Without parentheses:
  1. Negation \( \neg \)
  2. Conjunction \( \land \)
  3. Disjunction \( \lor \)
  4. Implication \( \rightarrow \)

- Right-associative:
  \[ p \rightarrow q \rightarrow r \text{ means } p \rightarrow (q \rightarrow r) \]
Declarative Sentence

- Statement
  - True
  - False

- Simple
  - “The sun is shining.”
  - “Tomorrow is Wednesday.”
  - \( \rightarrow \) Propositional Atoms

- With Structure
  - “If today is Tuesday, tomorrow is Wednesday.”
    - \( p \rightarrow q \), \( p \): “Today is Tuesday.” \( q \): “Tomorrow is Wednesday.”
  - “This lecture is exciting and not boring.”
    - \( p \land \neg q \), \( p \): “This lecture is exciting.” \( q \): “This lecture is boring.”
Non-Declarative Sentences

- **Questions**
  - “What time is it?”

- **Commands**
  - “Do your homework!”

- **Exclamations**
  - “Oh my god!”

- **Various others**
  - “Ready, steady, go.”
  - “Good night, my friend.”
  - “May the force be with you.”
  - “Live long and prosper.”
Models

- Model $\cong$ Valuation $\cong$ Environment $\cong$ Interpretation $\cong$ Assignment
- Mapping: \{Atomic propositions\} $\mapsto$ \{True, False\}
Caveat: “Model”

Sometimes: Term “Model” only used, if it makes formula true!
Notation

- **True:**
  - $T$
  - 1
  - $\top$ (LaTeX: \top)

- **False:**
  - $F$
  - 0
  - $\bot$ (LaTeX: \bot)
Satisfiability

- (At least) one model satisfies formula
Satisfiability (cont.)

- One Model
- Several Models
- All Models
Validity

- All models satisfy formula
Semantic Entailment

- $\phi$ is special case of $\psi$
Semantic Entailment (cont.)

- **Written:** \[ \varphi \models \psi \] (Latex: \models)

- **Meaning:** \( M \models \varphi \Rightarrow M \models \psi \)
  - **Note:**
    \[ M \not\models \varphi \Rightarrow \begin{cases} M \models \psi, & \text{or} \\ M \not\models \psi \end{cases} \]

- **Examples:**
  - \( (p \land q) \models p \)
  - \( (p \lor q) \not\models p \)
Semantic Entailment (cont.)

$\varphi \models \psi$

$\varphi \not\models \psi$

$\varphi \not\models \psi$

$\bot \models \psi$
Semantic Equivalence

- Written: \( \varphi \equiv \psi \)
- Special Case of Semantic Entailment
  - \( \varphi \models \psi \), and
  - \( \psi \models \varphi \).
  - Thus: \( \varphi, \psi \) satisfied by the same models
Relation between these Notions

- Close relation between...
  - Satisfiability
  - Validity
  - Semantic Entailment
  - Semantic Equivalence

- Can be reduced to each other
  - Algorithm for one → Solve all

More on October 28, 2014
Truth Table

- Row for each Model $\mathcal{M}_i$
  - $\#\text{Rows} = 2^{\#\text{Vars}}$

- Column for each (sub-)formula $\varphi_j$

- Entry $E_{i,j}$
  - True, if $\mathcal{M}_i \models \varphi_j$
  - False, if $\mathcal{M}_i \not\models \varphi_j$
# Truth Table: Example

<table>
<thead>
<tr>
<th>#</th>
<th>p</th>
<th>q</th>
<th>r</th>
<th>( \neg r )</th>
<th>q ( \lor \neg r )</th>
<th>p ( \land (q \lor \neg r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>1</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>2</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
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<tr>
<td>3</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>4</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
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</tr>
<tr>
<td>5</td>
<td>T</td>
<td>F</td>
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<tr>
<td>6</td>
<td>T</td>
<td>T</td>
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<tr>
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</tbody>
</table>
## Truth Table: Example

<table>
<thead>
<tr>
<th>#</th>
<th>p</th>
<th>q</th>
<th>r</th>
<th>¬r</th>
<th>q ∨ ¬r</th>
<th>p ∧ (q ∨ ¬r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
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<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>
Truth Table: Usage

- **Satisfiability**
  - (At least) one row with True?

- **Validity**
  - All rows True?

- **Entailment** $\varphi \vdash \psi$
  - $\psi$ has True at least where $\varphi$ has True?

- **Equivalence** $\varphi \equiv \psi$
  - $\varphi, \psi$ have True in same rows?
Applications

- Choose “Classical Question”
  - Often: Satisfiability

- Find Encoding
  - Variables
  - Meaning

- Construct Formula
  - Answer question

- Interpret Result
Application: Sudoku

- One number (1-9) in each square
- All numbers in one row
- All numbers in one column
- All numbers in one 3x3 square must be different
- Usually: Some numbers given
Summary

- Syntax
  - Elements/Symbols
  - Grammar
- Declarative Sentences
- Semantics
  - Models
  - Recursive Definition
  - Truth Tables
- Satisfiability, Validity, Semantic Entailment & Equivalence
- Example Applications